As we are interested on the probability as $n \to \infty$, we may only consider the case where $n \geq 3$. Let $Q_n$ be the probability such that for each pair of vertices $v$ and $w$, there exists a common neighbor $u$. Such a graph is clearly connected, so we have $Q_n \leq P_n \leq 1$. Given two vertices $v$ and $w$, the probability of a vertex $u$ which is not $v$ nor $w$ being a common neighbor of $v$ and $w$ is $\frac{1}{9}$. As the edges are added independently, the probability of $v$ and $w$ not having a common neighbor is $\left(\frac{8}{9}\right)^{n-2}$. Using the union bound, the probability of the existence of two vertices having no common neighbors is bounded above by $\left(\frac{n}{2}\right) \left(\frac{8}{9}\right)^{n-2} = \frac{n^2 - n}{2} \left(\frac{8}{9}\right)^{n-2}$.

Using L’Hôpital’s rule twice we have

$$
\lim_{n \to \infty} \frac{n^2 - n}{2}\left(\frac{8}{9}\right)^{n-2} = \lim_{n \to \infty} \frac{n^2 - n}{2 \left(\frac{9}{8}\right)^{n-2}} = \lim_{n \to \infty} \frac{2n - 1}{2 \ln(9/8) \left(\frac{9}{8}\right)^{n-2}} = \lim_{n \to \infty} \frac{2}{2 \left(\ln(9/8)\right)^{2} \left(9/8\right)^{n-2}} = 0.
$$

Meanwhile by the definition of $Q_n$, the probability of the existence of two vertices having no common neighbors is $1 - Q_n$. Thus by Sandwich theorem we have $\lim_{n \to \infty} (1 - Q_n) = 0$, that is, $Q_n \to 1$ as $n \to \infty$. Again by Sandwich theorem, we conclude that $\lim_{n \to \infty} P_n = 1$. 
