POW 2020-14

2016____ Chae Jiseok

September 14, 2020

As we are interested on the probability as $n \to \infty$, we may only consider the case where $n \geqslant 3$. Let Q_n be the probability such that for each pair of vertices v and w, there exists a common neighbor u. Such a graph is clearly connected, so we have $Q_n \leqslant P_n \leqslant 1$. Given two vertices v and w, the probability of a vertex u which is not v nor w being a common neighbor of v and w is $\frac{1}{9}$. As the edges are added independently, the probability of v and w not having a common neighbor is $\left(\frac{8}{9}\right)^{n-2}$. Using the union bound, the probability of the existence of two vertices having no common neighbors is bounded above by $\binom{n}{2}\left(\frac{8}{9}\right)^{n-2} = \frac{n^2-n}{2}\left(\frac{8}{9}\right)^{n-2}$. Using L'Hôpital's rule twice we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2-n}{2}\left(\frac{8}{9}\right)^{n-2} &= \lim_{n\to\infty} \frac{n^2-n}{2\,(9/8)^{n-2}} \\
&= \lim_{n\to\infty} \frac{2n-1}{2\ln(9/8)\,(9/8)^{n-2}} \\
&= \lim_{n\to\infty} \frac{2}{2\,(\ln(9/8))^2\;(9/8)^{n-2}} = 0.
\end{aligned}$$

Meanwhile by the definition of Q_n, the probability of the existence of two vertices having no common neighbors is $1 - Q_n$. Thus by Sandwich theorem we have $\lim_{n\to\infty}(1-Q_n) = 0$, that is, $Q_n \to 1$ as $n \to \infty$. Again by Sandwich theorem, we conclude that $\lim_{n\to\infty} P_n = 1$.