POW 2020-12

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Consider that the tournament gives $+1$ point to the winner, -1 point to the loser, and 0 points to each player if the game is a draw. Instead of the point system given in the problem, we may show that there exists two players with the same point if at least 3/4 of the games are draws in the new system. To see that these two settings are equivalent, starting from the original point system, say that the points assigned to each games are doubled and every player gets $(n-1)$ points deducted after all games are played.

Following the new point system, all players have integer points. Also, drawn games do not change the points of the players, so by assumption at most 1/4 of the games could have changed the points the players have. For convenience we may call such games as *score-changing* games.

We seek for an upper bound for the number of score-changing games required in a tournament so that each players can have distinct total scores. A simple observation is that if some player's total score is k where $k > 0$, then that player should have won at least k games. So the sum of the scores of the players with positive scores is a lower bound for the number of score-changing games, because each score-changing game has one winner and one loser. Similarly, a player with total score $-k$ for some $k > 0$ should have lost at least k games, so the sum of the absolute values of the scores of the players with negative score is also a lower bound for the number of score-changing games.

We must have $n \geqslant 2$. First, let n be odd, so that $n = 2m+1$ for some integer $m > 0$. If there are at most m players with negative score, at most one player can have zero score, so there are at least m players with postive score. Since each of the positive scores must be distinct, and the minimum of the sum of m distinct positive integers is $\frac{m(m+1)}{2}$, we conclude that there should have been at least $\frac{m(m+1)}{2}$ score-changing games in this case. Conversely, if there are at least m players with negative score, then also because each of the m negative scores must be distinct there should be at least $\frac{m(m+1)}{2}$ score-changing games in this case. Thus if there are $2m+1$ players in the tournament with all distinct scores then there must be at least $\frac{m(m+1)}{2}$ score-changing games. However in a tournament with $(2m+1)$ players there are $m(2m+1)$ games played, so there can be at most $\frac{m(m+1/2)}{2}$ score-changing games. This leads to a contradiction, thus if there are $(2m+1)$ players then there must be two players with the same final score.

Now let n be even so that $n = 2m$ for some integer $m > 0$. If there are at most $(m-1)$ players with negative score then there can be at most one player with zero score thus there must be at least m players with positive score. Conversely, if there are more than $(m-1)$ players with negative score, this means that at least m players have negative score. Either case, as seen in the previous paragraph, asserts that there exists at least $\frac{m(m+1)}{2}$ score-changing games. However in a tournament with $2m$ players there are $m(2m-1)$ games played, so there can be at most $\frac{m(m-1/2)}{2}$ score-changing games. This leads to a contradiction, thus if there are $2m$ players then there must be two players with the same final score.

Therefore in any case, if at least 3/4 of the games were drawn in a tournament then there must exist two players with the same final score.