POW 2020-11

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June 20, 2020

Let $S = \{a, b\}$ and $F = F(S)$ be a free group on the set S so that it is of rank 2. Let H be a subgroup of F where $H = \left\langle \left\{ a^k b^k : k \in \mathbb{Z} \right\} \right\rangle$. Let $T = \left\{ a^k b^k : k \in \mathbb{Z} \right\} \cup \left\{ (a^k b^k)^{-1} : k \in \mathbb{Z} \right\}$ then every element in H can be written in a finite product of elements in T.

We show that H is not finitely generated despite being a subgroup of F. For the sake of contradiction, suppose that $H = \langle s_1, s_2, \cdots, s_m \rangle$ for some positive integer m and $s_1, \cdots, s_m \in H$. Then because $s_1, \cdots, s_m$ can each be written as a finite product of elements in T, there exists a finite subset T_0 of T such that $s_1, \cdots, s_m$ are all generated by elements in T_0, which implies $H = \langle T_0 \rangle$. Thus for some positive integer ℓ such that $T_0 \subset \left\{ a^k b^k : |k| \leqslant \ell \right\} \cup \left\{ (a^k b^k)^{-1} : |k| \leqslant \ell \right\}$ we shall have

$$\begin{aligned} H &= \langle T_0 \rangle \\ &\leqslant \left\langle \left\{ a^k b^k : |k| \leqslant \ell \right\} \cup \left\{ (a^k b^k)^{-1} : |k| \leqslant \ell \right\} \right\rangle \\ &= \left\langle \left\{ a^k b^k : |k| \leqslant \ell \right\} \right\rangle \\ &\leqslant \left\langle \left\{ a^k b^k : k \in \mathbb{Z} \right\} \right\rangle = H \end{aligned}$$

hence $H = \left\langle \left\{ a^k b^k : |k| \leqslant \ell \right\} \right\rangle$. This asserts that every element in H should be possible to be represented as a product of elements in $T_\ell = \left\{ a^k b^k : |k| \leqslant \ell \right\} \cup \left\{ b^{-k} a^{-k} = (a^k b^k)^{-1} : |-k| = |k| \leqslant \ell \right\}$. Now write $h \in H$ as in the reduced form $h = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots a^{\alpha_k} b^{\beta_k} \in H$ where $h = e_H$ which is the case where $k = 0$ and for convenience we set $\alpha_1 = 0$ in this case, or we allow only α_1 or β_k to be 0. Observe that h cannot be in the form of a^{α_1} or b^{β_1} for nonzero α_1 or β_1 since a product of elements in T_ℓ must have the property that the sum of exponents of a and the sum of exponents of b must be equal. By multiplying an element of T_ℓ to h it falls into one of the three following cases : α_1 remains unchanged, or everything is cancelled out so that $h = e_H$ and α_1 becomes 0, or $h = e_H$ so that α_1 is changed to some number with absolute number less than ℓ. That is, $|\alpha_1| \leqslant \ell$ for any $h \in H$. However if so then we must have $a^{\ell+1} b^{\ell+1} \notin H$, which is absurd. We conclude that our assumption that H is finitely generated must be false, that is, H is indeed not finitely generated.