Let's define a $n \times n$ matrix $A_n = (a_{ij})$ by $a_{ij} = x^{|i-j|}$ where x is a variable. If we let $p_n(x)$ denote the determinant of A_n then

$$p_n(x) = \sum_{\pi} sgn(\pi) \prod_{i=1}^{n} a_{i\pi(i)} = \sum_{\pi} sgn(\pi) \prod_{i=1}^{n} x^{|i-\pi(i)|} = \sum_{\pi} sgn(\pi) x^{d(\pi)}$$

where $d(\pi)$ is the displacement of π. So the number of even permutations with displacement d minus the number of odd permutations with displacement d is the coefficient of x^d in $p_n(x)$.

For $n > 1$,

$$\begin{vmatrix} 1 & x & \cdot & \cdot & x^{n-2} & x^{n-1} \\ x & 1 & \cdot & \cdot & x^{n-3} & x^{n-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ x^{n-2} & x^{n-3} & \cdot & \cdot & 1 & x \\ x^{n-1} & x^{n-2} & \cdot & \cdot & x & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & \cdot & \cdot & x^{n-2} & x^{n-1} \\ x & 1 & \cdot & \cdot & x^{n-3} & x^{n-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ x^{n-2} & x^{n-3} & \cdot & \cdot & 1 & x \\ 0 & 0 & \cdot & \cdot & 0 & 1-x^2 \end{vmatrix}$$

so that $p_n(x) = (1-x^2)p_{n-1}(x)$. (The equality is attained by subtracting the $(n-1)$th row times x from the nth row.) $p_1(x) = 1$ so we can inductively show that $p_n(x) = (1-x^2)^{n-1}$ and the coefficient of x^{2k} in $p_n(x)$ is $(-1)^k \binom{n-1}{k}$.