We will prove G is finitely generated with only the assumption that X is connected instead of being proper and geodesic.

First we note that every $g \in G$ is a homeomorphism. For $H \subset G, A \subset X$ we define $HA = \{ha | h \in H, a \in A\}$. If A is open then HA is open for any subset H of G. Let's call a subset A of X saturated if $A = GA$. Then GA is saturated for any $A \subset X$. If there is a cover by saturated open sets of X then there is a finite subcover due to the compactness of X/G. If we define $B(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\}, \overline{B}(x, \epsilon) = \{y \in X | d(x, y) \leq \epsilon\}$ for $x \in X, \epsilon > 0$ then for any $g \in G$, $g(\overline{B}(x, \epsilon)) = \overline{B}(gx, \epsilon)$ and $g(B(x, \epsilon)) = B(gx, \epsilon)$.
The crucial point we will prove is that X is locally compact which is that for any $x \in X$, there is a compact subspace of X that contains an open neighborhood of x. Let x be an arbitrary point of X. $\{x\}$ is compact so $G_x = \{g \in G | gx = x\}$ is finite.

Step 1. There is an $\epsilon > 0$ such that $g \in G_x$ if $d(gx, x) < \epsilon$
If for any $\epsilon > 0$ there exists a $g \in G$ such that $0 < d(gx, x) < \epsilon$, then there is a sequence $\{g_n\}_{n \geq 1}$ such that $\{g_n x\}_{n \geq 1}$ converges to x (and $g_n x \neq x$ for every n). Then $A = \{x\} \cup \{g_n x | n \geq 1\}$ is compact since any open neighborhood of x contains all but finitely many $g_n x$ which is a contradiction because $g_n A \cap A \neq \emptyset$ for every n. So there is an $\epsilon > 0$ such that $g \in G_x$ if $d(gx, x) < \epsilon$. Let $C_x = \overline{B}(x, \frac{\epsilon}{3})$.

Step 2. GC_x is closed

For any $g_1, g_2 \in G$, $d(g_1x, g_2x) = d(x, g_1^{-1}g_2x)$ so if the distance between g_1x and g_2x is smaller than ϵ they are equal. $GC_x = \bigcup_{g\in G} \overline{B}(gx, \frac{\epsilon}{3})$ is closed because for any $y \notin GC_x$ there is at most one gx such that $d(gx, y) < \frac{\epsilon}{2}$ (g may not be unique) so there is an open neighborhood of y disjoint from GC_x.

Step 3. X is locally compact

It suffices to prove that $C_x = \overline{B}(x, \frac{\epsilon}{3})$ is compact. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of C_x and let $G_x = \{g_1, g_2, \cdots, g_n\}$. The collection $\{V_\beta\}_{\beta\in J}$ of open sets of the form $B(x, \frac{\epsilon}{2}) \cap \bigcap_{i=1}^{n} g_i^{-1}U_i$ ($U_i \in \{U_\alpha\}$) covers C_x because for any $y \in C_x$ and i there is a U_i containing $g_iy \in C_x$ so $y \in g_i^{-1}U_i$. Then $\{GV_\beta | \beta \in J\}$ is an open cover of GC_x and $\{X - GC_x\} \cup \{GV_\beta | \beta \in J\}$ is an open cover of X of which every element is saturated. So there is a finite subcover $\{X - GC_x, GV_1, GV_2, \cdots, GV_m\}$ of X.

For any $y \in C_x$ there is a GV_j containing y. $V_j \subset B(x, \frac{\epsilon}{2})$ so for $g \in G$, $gV_j \subset B(gx, \frac{\epsilon}{2})$ and if $g \notin G_x$ then $y \notin gV_j$ because $d(x, y) \leq \frac{\epsilon}{3}, d(gx, x) \geq \epsilon$ implies $d(gx, y) > \frac{\epsilon}{2}$. So there is a $g_k \in G_x$ such that $y \in g_kV_j$ and if $V_j = B(x, \frac{\epsilon}{2}) \cap \bigcap_{i=1}^{n} g_i^{-1}U_i$ ($U_i \in \{U_\alpha\}$) then $g_k^{-1}y \in V_j \subset g_k^{-1}U_k, y \in U_k$. This shows that if we collect the associated U_i of $V_1, V_2, \cdots, V_m$ we get a open cover of C_x with (at most) mn elements which is a subcover of $\{U_\alpha\}$ and therefore C_x is compact.

Step 4. G is finitely generated

For every $x \in X$ pick an open neighborhood U_x contained in a compact set (for example $B(x, \frac{\epsilon}{3}) \subset C_x$ from above). $\{GU_x | x \in X\}$ is a saturated open cover of X so there is a finite subcover $\{GU_1, GU_2, \cdots, GU_l\}$. Then $U = \bigcup_{i=1}^{l} U_i$ is an open set contained in a compact set (because a finite union of compact sets is compact) such that $GU = X$. $G_0 = \{g \in G | gU \cap U \neq \emptyset\}$ is finite because U is contained in a compact set. Let G_1 be the subgroup of G generated by the set G_0. If $gU \cap g_1 U \neq \emptyset$ for $g \in G, g_1 \in G_1$ then $g_1^{-1} gU \cap U \neq \emptyset$ and $g_1^{-1} g \in G_0$ so $g \in G_1$. If $G_1 \neq G$ then $G_1 U$ and $(G - G_1)U$ are disjoint non-empty open sets and their union is X which is a contradiction so G is generated by the finite set G_0.