Proposition 1.1 Suppose that $x, y, z$ are positive integers satisfying
\[ 0 \leq x^2 + y^2 - xyz \leq z + 1. \] (1)
Then, $x^2 + y^2 - xyz$ is a perfect square.

Proof. We will first prove for $x = 1$. By substituting $x = 1$, we have
\[-1 \leq y^2 - yz = y(y - z) \leq z.\]
If $y < z$, then $y(y - z) = -1$, which implies $y = 1, z = 2$. Thus we have $x^2 + y^2 - xyz = 0$ which is a perfect square.
If $y = z$, then $x^2 + y^2 - xyz = 1$ which is a perfect square.
If $y > z$, then $z \leq z(y - z) < y(y - z) \leq z$, which is a contradiction.
Since the expression is symmetric, we can apply the same logic for $y = 1$.

Suppose proposition 1.1 holds for any $x, y$ such that $x \leq m$ and $y \leq n$ for positive integers $m$ and $n$. We want to show proposition 1.1 also holds for $x = m + 1$ and $y = n + 1$. For convenience, we will denote $m + 1$ and $n + 1$ by $x$ and $y$, respectively.
We may assume $x \leq y$ without loss of generality. Evaluate (1) with respect to $z$, and we get the following inequality:
\[ \frac{x^2 + y^2 - 1}{xy + 1} \leq z \leq \frac{x^2 + y^2}{xy}. \] (2)

Lemma 1. Suppose $x \geq 2$ and $y \geq 2$ satisfy (2). Then,
\[ z = \left\lfloor \frac{x^2 + y^2}{xy} \right\rfloor = \left\lfloor \frac{y}{x} + \frac{x}{y} \right\rfloor. \]

To show lemma 1.1, it is enough to show
\[ \frac{x^2 + y^2 - 1}{xy + 1} > \frac{x^2 + y^2}{xy} - 1. \]
This inequality is equivalent to $x^2y^2 > x^2 + y^2$ and it is easy to show since

$$x^2y^2 \geq 4y^2 > x^2 + y^2.$$  

By lemma 1.1 and since $x/y \leq 1$, we get

$$z = \left\lfloor \frac{y}{x} \right\rfloor \text{ or } \left\lfloor \frac{y}{x} \right\rfloor + 1.$$

i) $z = \lfloor y/x \rfloor$

This is equivalent to

$$\frac{y}{x} - 1 < z \leq \frac{y}{x}$$

$$\iff y - x < xz \leq y$$

$$\iff xz \leq y < xz + x$$

$$\iff y = xz + w, \ 0 \leq w < x$$  \hspace{1cm} (3)

Substituting with (3), we have

$$z + 1 \geq x^2 + y^2 - xyz$$

$$= x^2 + (xz)^2 + 2xwz + w^2 - (xz)^2 - xwz$$

$$= x^2 + w^2 + xwz$$

The first inequality comes from (1). Since we have $x \geq 2$, this inequality implies $z+1 \geq 4+2wz$. If $w \geq 1$, this gives $z+1 \geq 4+2z$, so $z \leq -3$. Contradiction. Thus we can conclude $w = 0$ and $y = xz$. This shows

$$x^2 + y^2 - xyz = x^2$$

which is obviously a perfect square.

ii) $z = \lfloor y/x \rfloor + 1$

This is equivalent to

$$\frac{y}{x} < z \leq \frac{y}{x} + 1$$

$$\iff y < xz \leq y + x$$
\[
\iffalse xz - x \leq y < xz 
\fi
\iffalse y = xz - w, \ 0 < w \leq x \quad (4)
\fi
\]

Substituting with (4), we have

\[
0 \leq x^2 + y^2 - xyz
= x^2 + (xz)^2 - 2xz + w^2 - (xz)^2 + xwz
= x^2 + w^2 - xwz
\leq z + 1
\quad (5)
\]

The first and last inequality comes from (1). If \(y = x\) or \(w = x\), then

\[
0 \leq x^2 + y^2 - xyz
= x^2 + w^2 - xwz
= 2x^2 - zx^2
= (2 - z)x^2
\]

which implies \(z = 1\) or \(z = 2\). In each case, \(x^2 + y^2 - xyz = x^2\) and \(x^2 + y^2 - xyz = 0\), which are perfect squares. Otherwise, \(x < y = n + 1\) and \(w < x = m + 1\) by (4). By induction hypothesis and (5), \(x^2 + y^2 - xyz = w^2 + x^2 - wxz\) is a perfect square.

\[
\square
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