POW 2020-07

Perfect Square

2016_____ 임상호

Proposition 1.1 Suppose that x, y, z are positive integers satisfying

$$0 \leq x^2 + y^2 - xyz \leq z + 1. \tag{1}$$

Then, $x^2 + y^2 - xyz$ is a perfect square.

Proof. We will first prove for $x = 1$. By substituting $x = 1$, we have

$$-1 \leq y^2 - yz = y(y - z) \leq z.$$

If $y < z$, then $y(y-z) = -1$, which implies $y = 1, z = 2$. Thus we have $x^2 + y^2 - xyz = 0$ which is a perfect square.
If $y = z$, then $x^2 + y^2 - xyz = 1$ which is a perfect square.
If $y > z$, then $z \leq z(y - z) < y(y - z) \leq z$, which is a contradiction.
Since the expression is symmetric, we can apply the same logic for $y = 1$.

Suppose proposition 1.1 holds for any x, y such that $x \leq m$ and $y \leq n$ for positive integers m and n. We want to show proposition 1.1 also holds for $x = m + 1$ and $y = n + 1$. For convenience, we will denote $m + 1$ and $n + 1$ by x and y, respectively.
We may assume $x \leq y$ without loss of generality. Evaluate (1) with respect to z, and we get the following inequality:

$$\frac{x^2 + y^2 - 1}{xy + 1} \leq z \leq \frac{x^2 + y^2}{xy}. \tag{2}$$

Lemma 1. *Suppose $x \geq 2$ and $y \geq 2$ satisfy (2). Then,*

$$z = \left\lfloor \frac{x^2 + y^2}{xy} \right\rfloor = \left\lfloor \frac{y}{x} + \frac{x}{y} \right\rfloor.$$

To show lemma 1.1, it is enough to show

$$\frac{x^2 + y^2 - 1}{xy + 1} > \frac{x^2 + y^2}{xy} - 1.$$

This inequality is equivalent to $x^2y^2 > x^2 + y^2$ and it is easy to show since

$$x^2y^2 \geq 4y^2 > x^2 + y^2.$$

By lemma 1.1 and since $x/y \leq 1$, we get

$$z = \left\lfloor \frac{y}{x} \right\rfloor \text{ or } \left\lfloor \frac{y}{x} \right\rfloor + 1.$$

i) $z = \lfloor y/x \rfloor$

This is equivalent to

$$\begin{aligned} & \frac{y}{x} - 1 < z \leq \frac{y}{x} \\ \iff \quad & y - x < xz \leq y \\ \iff \quad & xz \leq y < xz + x \\ \iff \quad & y = xz + w,\ 0 \leq w < x \end{aligned} \tag{3}$$

Substituting with (3), we have

$$\begin{aligned} z + 1 &\geq x^2 + y^2 - xyz \\ &= x^2 + (xz)^2 + 2xwz + w^2 - (xz)^2 - xwz \\ &= x^2 + w^2 + xwz \end{aligned}$$

The first inequality comes from (1). Since we have $x \geq 2$, this inequality implies $z+1 \geq 4+2wz$. If $w \geq 1$, this gives $z+1 \geq 4+2z$, so $z \leq -3$. Contradiction. Thus we can conclude $w = 0$ and $y = xz$. This shows

$$x^2 + y^2 - xyz = x^2$$

which is obviously a perfect square.

ii) $z = \lfloor y/x \rfloor + 1$

This is equivalent to

$$\begin{aligned} & \frac{y}{x} < z \leq \frac{y}{x} + 1 \\ \iff \quad & y < xz \leq y + x \end{aligned}$$

$$
\begin{aligned}
&\iff \quad xz - x \le y < xz \\
&\iff \quad y = xz - w, \ 0 < w \le x \qquad (4)
\end{aligned}
$$

Substituting with (4), we have

$$
\begin{aligned}
0 &\le x^2 + y^2 - xyz \\
&= x^2 + (xz)^2 - 2xwz + w^2 - (xz)^2 + xwz \\
&= x^2 + w^2 - xwz \\
&\le z + 1 \qquad (5)
\end{aligned}
$$

The first and last inequality comes from (1). If $y = x$ or $w = x$, then

$$
\begin{aligned}
0 &\le x^2 + y^2 - xyz \\
&= x^2 + w^2 - xwz \\
&= 2x^2 - zx^2 \\
&= (2 - z)x^2
\end{aligned}
$$

which implies $z = 1$ or $z = 2$. In each case, $x^2 + y^2 - xyz = x^2$ and $x^2 + y^2 - xyz = 0$, which are perfect squares. Otherwise, $x < y = n + 1$ and $w < x = m + 1$ by (4). By induction hypothesis and (5), $x^2 + y^2 - xyz = w^2 + x^2 - wxz$ is a perfect square.

□