POW 2020-05

2018 김기수

1 Construct Y

We will construct a “completion” of X, and imbed X to its completion by inclusion-like mapping. Let S_X be a collection of Cauchy sequences in X. Define a relation $\sim$ on S_X by

$$\{x_n\} \sim \{y_n\} \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

This is an equivalence relation. The reflexivity and symmetry hold since $d(x,x) = 0$ and $d(x,y) = d(y,x)$. Suppose $\{x_n\}, \{y_n\}, \{z_n\}$ are Cauchy sequences in X such that $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$. Observe that $\lim_{n\to\infty} d(x_n, z_n) \le \lim_{n\to\infty} (d(x_n, y_n) + d(y_n, z_n)) = \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) = 0$ hence $\{x_n\} \sim \{z_n\}$ and the transitivity holds. Let $Y = S_X/\sim$ be the collection of equivalence classes of S_X.

Define a function $d' : Y \times Y \to \mathbb{R}$ by

$$([\{x_n\}], [\{y_n\}]) \mapsto \lim_{n\to\infty} d(x_n, y_n).$$

Claim that d' is well defined, and is a metric on Y.

Firstly, we will show that $\{d(x_n, y_n)\}$ converges for $\{x_n\}, \{y_n\} \in S_X$. For arbitrary $\varepsilon > 0$, there exists an integer N such that $n \ge N$ implies $d(x_N, x_n) \le \varepsilon/2$ and $d(y_N, y_n) \le \varepsilon/2$, since the sequences are Cauchy. By the triangle inequality, we have $d(x_n, y_n) \le d(x_n, x_N) + d(x_N, y_N) + d(y_N, y_n) \le d(x_N, y_N) + \varepsilon$. Similarly, $d(x_N, y_N) \le d(x_n, y_n) + \varepsilon$. Deduce that $d(x_n, y_n) \in [d(x_N, y_N) - \varepsilon, d(x_N, y_N) + \varepsilon]$ if $n \ge N$. For $\varepsilon_i = \frac{1}{2^i}$, choose $N_1 \le N_2 \le \cdots$ such that $d(x_n, y_n) \in [d(x_{N_i}, y_{N_i}) - \varepsilon_i, d(x_{N_i}, y_{N_i}) + \varepsilon_i]$ if $n \ge N_i$, as the above argument. Let $\alpha_i = d(x_{N_i}, y_{N_i})$ and $A_0 = \mathbb{R}, A_i = [\alpha_i - \varepsilon_i, \alpha_i + \varepsilon_i] \cap A_{i-1}$. For arbitrary n, $\bigcap_{i=1}^{n} A_i$ is nonempty since $\alpha_{n+1} \in A_i$ for $1 \le i \le n$. The collection of closed sets $\{A_i\}_{i\ge1}$ has the finite intersection property as $A_1 \supset A_2 \supset \cdots$. Using the compactness of A_1, deduce that $A = \bigcap_{i=1}^{\infty} A_i$ is nonempty. This set cannot contain more than one element. On contrary, assume that $a < b$ are in

A. There exists ε_n such that $2\varepsilon_n < b - a$. It contradicts $A \subset A_n$ hence the assumption is absurd. Let $A = \{\alpha\}$. α is the limit of $\{d(x_n, y_n)\}$. For $\varepsilon > 0$, choose $\varepsilon_i = \frac{1}{2^i}$ so that $3\varepsilon_i < \varepsilon$. α is contained in A_i, and for every $n \geq N_i$ $d(x_n, y_n) \in A_i$. Hence $|\alpha - d(x_n, y_n)| \leq 2\varepsilon_i < \varepsilon$ for every $n \geq N_i$.

Let $\{x'_n\}$ be an another representative of $[\{x_n\}]$, and $\{y_n\}$ be a Cauchy sequence. Observe that

$$d(x'_n, y_n) \leq d(x'_n, x_n) + d(x_n, y_n)$$

hence

$$\lim_{n\to\infty} d(x'_n, y_n) \leq \lim_{n\to\infty} d(x'_n, x_n) + \lim_{n\to\infty} d(x_n, y_n) = 0 + \lim_{n\to\infty} d(x_n, y_n).$$

Similarly, $\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x'_n, y_n)$ holds so two limits are the same. This implies that d' is independent of the choice of representatives.

It is easy to verify that d' is a metric on Y. Let s, t, u be elements of Y, and $\{a_n\}, \{b_n\}, \{c_n\}$ be their representatives. Since $d(x, y) \geq 0$ for every $x, y \in X$, $d'(s, t) \geq 0$. By the definition of $Y = S_X / \sim$, $d'(s, t) = 0$ if and only if $s = t$. Clearly $d'(s, t) = d'(t, s)$. To obtain the triangle inequality, observe that

$$\begin{aligned}
d'(s, u) &= \lim_{n\to\infty} d(a_n, c_n) \\
&\leq \lim_{n\to\infty} (d(a_n, b_n) + d(b_n, c_n)) \\
&= \lim_{n\to\infty} d(a_n, b_n) + \lim_{n\to\infty} d(b_n, c_n) \\
&= d'(s, t) + d'(t, u).
\end{aligned}$$

Therefore, (Y, d') is a metric space. Let rename Y as X^*.

2 Completeness of $Y = X^*$

Firstly, observe that (X, d) can be isometrically imbedded into X^* naturally. For $x \in X$, consider the sequence $\{a_n\}$ such that every term is equal to x. Note that this is a Cauchy sequence. Let $x^* = [\{a_n\}]$. Define a map $\iota : X \to X^*, x \mapsto x^*$. Because $d'(\iota(x), \iota(y)) = d'(x^*, y^*) = \lim_{n\to\infty} d(x, y) = d(x, y)$, ι is an isometric embedding.

$\iota(X)$ is dense in X^*. Let $[\{x_n\}] \in X^*$. Since $\{x_n\}$ is Cauchy, for $\varepsilon > 0$, there exists N such that $d(x_N, x_n) \leq \varepsilon/2$ if $n \geq N$. The ε-ball centered at $[\{x_n\}]$ intersects $\iota(X)$: $d'(x^*_N, [\{x_n\}]) = \lim_{n\to\infty} d(x_N, x_n) \leq \varepsilon/2 < \varepsilon$. This argument can be applied for arbitrary ε hence $[\{x_n\}]$ is a limit point of $\iota(X)$.

We will show that (Y, d') is a complete metric space. Let $y_1, y_2, \cdots$ be a Cauchy sequence in X^*, and let $y_i = [\{a_{ij}\}]$. Let $\varepsilon_i = \frac{1}{2^i}$, and choose $N_1 \leq N_2 \leq \cdots$ such that $d'(a^*_{iN_i}, y_i) \leq \frac{\varepsilon_i}{5}$ hence $d(a_{iN_i}, a_{in}) \leq \frac{\varepsilon_i}{4}$ for every sufficiently large n. As observed in the above paragraph, these choices are possible. Claim that the sequence $\{a_{nN_n}\}$ is Cauchy, and the limit of the sequence $\{y_i\}$.

For $\varepsilon > 0$, fix k such that $\varepsilon_k < \varepsilon$. Choose $M \geq k$ such that $d'(y_i, y_j) \leq \frac{\varepsilon_k}{4}$ if $i, j \geq M$. Observe that

$$\begin{aligned} d(a_{iN_i}, a_{jN_j}) &\leq d(a_{iN_i}, a_{in}) + d(a_{in}, a_{jn}) + d(a_{jn}, a_{jN_j}) \\ &\leq \frac{\varepsilon_i}{4} + d(a_{in}, a_{jn}) + \frac{\varepsilon_j}{4} \\ &\leq \frac{\varepsilon_k}{2} + d(a_{in}, a_{jn}) \end{aligned}$$

for $M \leq i < j$ and for sufficiently large $n \geq N_j$. Since $\lim_{n\to\infty} d(a_{in}, a_{jn}) = d'(y_i, y_j) \leq \frac{\varepsilon_k}{4}$, there exists sufficiently large n such that $d(a_{in}, a_{jn}) \leq \frac{\varepsilon_k}{2}$. Therefore, $d(a_{iN_i}, a_{jN_j}) \leq \frac{\varepsilon_k}{2} + \frac{\varepsilon_k}{2} = \varepsilon_k < \varepsilon$ for every $i, j \geq M$ hence $\{a_{iN_i}\}$ is Cauchy and $[\{a_{iN_i}\}] \in X^*$.

Finally, it is remianing to show $\lim_{i\to\infty} y_i = [\{a_{nN_n}\}]$. Let $y = [\{a_{nN_n}\}]$. Claim that $\lim_{i\to\infty} d'(y_i, y) = 0$. This is equivalent to show that $\lim_{i\to\infty} \lim_{n\to\infty} d(a_{in}, a_{nN_n}) = 0$. For given $\varepsilon > 0$, fix k such that $d'(y_i, y_j) \leq \frac{\varepsilon}{4}$ if $i, j \geq k$, and that $\varepsilon_k = \frac{1}{2^k} < \frac{\varepsilon}{4}$. For chosen k, choose $l \geq k$ such that $i, j \geq l$ implies $d'(y_i, y_j) \leq \frac{\varepsilon_k}{4}$. Assume that $n \geq N_l$ and $n \geq l$. For $m \geq N_n$, observe that

$$\begin{aligned} d(a_{ln}, a_{nN_n}) &\leq d(a_{ln}, a_{lN_l}) + d(a_{lN_l}, a_{nN_n}) \\ &\leq d(a_{ln}, a_{lN_l}) + d(a_{lN_l}, a_{lm}) + d(a_{lm}, a_{nm}) + d(a_{nm}, a_{nN_n}) \\ &\leq \frac{\varepsilon_l}{4} + \frac{\varepsilon_k}{4} + d(a_{lm}, a_{nm}) + \frac{\varepsilon_k}{4} \\ &\leq \frac{\varepsilon_l}{4} + \frac{\varepsilon_k}{2} + d(a_{lm}, a_{nm}). \end{aligned}$$

By our choice of l, $d(a_{lm}, a_{nm}) \leq \frac{\varepsilon_k}{2}$ for sufficiently large m. Hence $d(a_{ln}, a_{nN_n}) \leq \frac{\varepsilon_l}{4} + \frac{\varepsilon_k}{2} + \frac{\varepsilon_k}{2} \leq 2\varepsilon_k$ for every sufficiently large n. Assume $i \geq k$. Observe that

$$d(a_{in}, a_{nN_n}) \leq d(a_{in}, a_{ln}) + d(a_{ln}, a_{nN_n}) \leq \frac{\varepsilon}{3} + 2\varepsilon_k < \frac{5}{6}\varepsilon,$$

for every large n such that $d(a_{in}, a_{ln}) \leq \frac{\varepsilon}{3}$. This inequality holds by the choice of k and $i, l \geq k$. Deduce that $\lim_{n\to\infty} d(a_{in}, a_{nN_n}) \leq \frac{5}{6}\varepsilon < \varepsilon$ for every $i \geq k$. This argument holds for arbitrary $\varepsilon > 0$ hence $\lim_{i\to\infty} \lim_{n\to\infty} d(a_{in}, a_{nN_n}) = 0$ and $\lim_{i\to\infty} d'(y_i, y) = 0$. Therefore, every Cauchy sequence in X^* converges.