Let's prove $\lim_{n\to\infty} g_n(y_0) = g(y_0)$.

For any $y_1, y_2 \in \mathbb{R}$ and $n = 1, 2, 3, \cdots$

$$\Big(\int_{-1}^{1} e^{y_1 f_n(x)} dx\Big)\Big(\int_{-1}^{1} e^{y_2 f_n(x)} dx\Big) \geq \Big(\int_{-1}^{1} e^{\frac{y_1+y_2}{2} f_n(x)} dx\Big)^2$$

by the Cauchy-Schwarz inequality.

$$\begin{aligned} g_n(y_1) + g_n(y_2) &= \log \int_{-1}^{1} e^{y_1 f_n(x)} dx + \log \int_{-1}^{1} e^{y_2 f_n(x)} dx \\ &\geq 2 \log \int_{-1}^{1} e^{\frac{y_1+y_2}{2} f_n(x)} dx = 2 g_n(\frac{y_1+y_2}{2}) \end{aligned}$$

Let $\epsilon > 0$ be fixed. There is an $\alpha > 0$ such that $g(y_0 - \alpha), g(y_0 + \alpha), g(y_0 + 2\alpha)$ are all contained in $\Big(g(y_0) - \frac{\epsilon}{6}, g(y_0) + \frac{\epsilon}{6}\Big)$ and an integer N such that $|g_n(y) - g(y)| < \frac{\epsilon}{6}$ for all $n > N$ and $y \in \{y_0 - \alpha, y_0 + \alpha, y_0 + 2\alpha\}$.

Then for all $n > N$,

$$g_n(y_0) \geq 2g_n(y_0 + \alpha) - g_n(y_0 + 2\alpha) > 2g(y_0 + \alpha) - g(y_0 + 2\alpha) - \frac{\epsilon}{2} > g(y_0) - \epsilon$$

$$g_n(y_0) \leq \frac{1}{2}\Big(g_n(y_0-\alpha)+g_n(y_0+\alpha)\Big) < \frac{1}{2}\Big(g(y_0-\alpha)+g(y_0+\alpha)+\frac{\epsilon}{3}\Big) < g(y_0)+\frac{\epsilon}{3}.$$

For any $\epsilon > 0$ there exists an integer N such that $|g_n(y_0) - g(y_0)| < \epsilon$ for all $n > N$ so $\lim_{n\to\infty} g_n(y_0) = g(y_0)$.