KAIST POW 2020-03

Chanjin You

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Problem. A permutation $\pi : [n] \to [n]$ is graceful if $|\pi(i+1) - \pi(i)| \neq |\pi(j+1) - \pi(j)|$ for all $i \neq j \in [n-1]$. For a graceful permutation $\pi : [2k+1] \to [2k+1]$ with $\pi(\{2, 4, \cdots, 2k\}) = [k]$, prove that $\pi(1) + \pi(2k+1) = 3k + 2$.

Solution. Since π is a permuation and $\pi(\{2, 4, \cdots, 2k\}) = [k]$, we have

$$\pi(\{1, 3, \cdots, 2k+1\}) = [2k+1] \setminus [k]. \tag{1}$$

For $i \in [2k]$,

$$|\pi(i+1) - \pi(i)| = \begin{cases} \pi(2m-1) - \pi(2m) & \text{if } i = 2m-1 \\ \pi(2m+1) - \pi(2m) & \text{if } i = 2m \end{cases}$$

where $m \in [k]$. One can easily check $1 \leq |\pi(i+1) - \pi(i)| \leq 2k$. Since π is graceful, $g : [2k] \to [2k]$ defined by $g(i) = |\pi(i+1) - \pi(i)|$ is a permutation. Therefore

$$\sum_{i=1}^{2k} |\pi(i+1) - \pi(i)| = \sum_{i=1}^{2k} i. \tag{2}$$

The left side of (2) is

$$\begin{aligned} &\sum_{m=1}^{k} (\pi(2m-1) - \pi(2m)) - \sum_{m=1}^{k} (\pi(2m+1) - \pi(2m)) \\ &= \pi(1) + \pi(2k+1) + 2\sum_{m=2}^{k} \pi(2m-1) - 2\sum_{m=1}^{k} \pi(2m) \\ &= \pi(1) + \pi(2k+1) + 2\sum_{m=2}^{k} \pi(2m-1) - 2\sum_{i=1}^{k} i. \end{aligned}$$

Therefore,

$$\pi(1) + \pi(2k+1) + 2\sum_{m=2}^{k} \pi(2m-1) = \sum_{i=1}^{2k} i + 2\sum_{i=1}^{k} i = k(3k+2). \tag{3}$$

On the other hand, (1) implies that

$$2(\pi(1) + \pi(2k+1)) + 2\sum_{m=2}^{k} \pi(2m-1) = 2\sum_{i=k+1}^{2k+1} i = (k+1)(3k+2). \tag{4}$$

From (3) and (4), we conclude that

$$\pi(1) + \pi(2k+1) = 3k + 2.$$

□