Problem. A permutation $\pi: [n] \to [n]$ is graceful if $|\pi(i + 1) - \pi(i)| \neq |\pi(j + 1) - \pi(j)|$ for all $i \neq j \in [n - 1]$. For a graceful permutation $\pi: [2k + 1] \to [2k + 1]$ with $\pi(\{2, 4, \cdots, 2k\}) = [k]$, prove that $\pi(1) + \pi(2k + 1) = 3k + 2$.

Solution. Since $\pi$ is a permutation and $\pi(\{2, 4, \cdots, 2k\}) = [k]$, we have

$$\pi(\{1, 3, \cdots, 2k + 1\}) = [2k + 1] \setminus [k]. \quad (1)$$

For $i \in [2k]$, $|\pi(i + 1) - \pi(i)| = \begin{cases} \pi(2m - 1) - \pi(2m) & \text{if } i = 2m - 1 \\ \pi(2m + 1) - \pi(2m) & \text{if } i = 2m \end{cases}$

where $m \in [k]$. One can easily check $1 \leq |\pi(i + 1) - \pi(i)| \leq 2k$. Since $\pi$ is graceful, $g: [2k] \to [2k]$ defined by $g(i) = |\pi(i + 1) - \pi(i)|$ is a permutation. Therefore

$$\sum_{i=1}^{2k} |\pi(i + 1) - \pi(i)| = \sum_{i=1}^{2k} i. \quad (2)$$

The left side of (2) is

$$\sum_{m=1}^{k} \left( \pi(2m - 1) - \pi(2m) \right) - \sum_{m=1}^{k} \left( \pi(2m + 1) - \pi(2m) \right)$$

$$= \pi(1) + \pi(2k + 1) + 2 \sum_{m=2}^{k} \pi(2m - 1) - 2 \sum_{m=1}^{k} \pi(2m)$$

$$= \pi(1) + \pi(2k + 1) + 2 \sum_{m=2}^{k} \pi(2m - 1) - 2 \sum_{i=1}^{k} i.$$

Therefore,

$$\pi(1) + \pi(2k + 1) + 2 \sum_{m=2}^{k} \pi(2m - 1) = \sum_{i=1}^{2k} i + \sum_{i=1}^{k} i = k(3k + 2). \quad (3)$$

On the other hand, (1) implies that

$$2(\pi(1) + \pi(2k + 1)) + 2 \sum_{m=2}^{k} \pi(2m - 1) = 2 \sum_{i=k+1}^{2k+1} i = (k + 1)(3k + 2). \quad (4)$$

From (3) and (4), we conclude that

$$\pi(1) + \pi(2k + 1) = 3k + 2.$$