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Suppose that n is odd. Then in S, there are odd number of odd numbers, hence the sum of all elements in S is odd. If there exists A and B satisfying the conditions, then sum of elements of S should be twice the sum of elements of A, so the sum of elements of S must be even. Therefore there cannot exist two subsets A and B of S satisfying the given conditions. Hence n cannot be odd.

Suppose $n = 2$, and let m be arbitrary. Suppose there exists A and B satisfying the given conditions. Without loss of generality, we may assume $(m+1)^2 \in A$. Then we have the following three cases.

- If $(m+2)^2 \in A$, then the sum of elements of A is $2m^2 + 6m + 5$, while the sum of elements of B is $2m^2 + 14m + 25$, which is strictly greater than that of A for $m \geqslant 0$.
- If $(m+3)^2 \in A$, then the sum of elements of A is $2m^2 + 8m + 10$, while the sum of elements of B is $2m^2 + 12m + 20$, which is strictly greater than that of A for $m \geqslant 0$.
- If $(m+4)^2 \in A$, then the sum of elements of A is $2m^2 + 10m + 17$, while the sum of elements of B is $2m^2 + 10m + 13$, which is strictly less than that of A for $m \geqslant 0$.

Therefore there cannot exist A and B satisfying the given conditions. Hence n cannot be 2.

Before we consider the case when $n \geqslant 2$ and n is even, we consider the following lemma.

Lemma 1. *Any integer $k \geqslant 2$ can be expressed as $k = 2a + 3b$, where a and b are nonnegative integers.*

Proof. It is clear that $2 = 2 \cdot 1 + 3 \cdot 0$, $3 = 2 \cdot 0 + 3 \cdot 1$, and $4 = 2 \cdot 2 + 3 \cdot 0$.

For $k \geqslant 5$, depending on $k \bmod 3$, there exists positive integers q and r such that $2 \leqslant r \leqslant 4$ and $k = 3q + r$. Expressing r as what we have seen in the previous paragraph, k is clearly representable in the form of $k = 2a + 3b$ with a, b being nonnegative integers. □

From this lemma we have an immediate corollary, as follows.

Corollary 1. *For any positive integer $k \geqslant 8$ with k divisible by 4, k can be represented in the form of $k = 8a + 12b$ for nonnegative integers a and b.*

Now suppose n is an even integer such that $n \geqslant 4$. Let m be arbitrary. Observe that, when $n = 4$, we have a partition of S as

$$A = \{(m+1)^2, (m+4)^2, (m+6)^2, (m+7)^2\}, \quad B = \{(m+2)^2, (m+3)^2, (m+5)^2, (m+8)^2\}$$

satisfying the given conditions. Also when $n = 6$, we have a partition of S as

$$A = \{(m+1)^2, (m+3)^2, (m+7)^2, (m+8)^2, (m+9)^2, (m+11)^2\},$$
$$B = \{(m+2)^2, (m+4)^2, (m+5)^2, (m+6)^2, (m+10)^2, (m+12)^2\}$$

satisfying the given conditions. Now if n is even and $n \geqslant 8$, then by **Corollary 1**, there exists positive integers α and β such that $2n = 8\alpha + 12\beta$. Thus we can divide S into α chuncks of 8 consecutive squares, and β chuncks of 12 consecutive squares, with all chuncks being disjoint. For each chunck, partition the squares as in the cases $n = 4$ and $n = 6$. Collecting each partitions from chuncks, we obtain a partition A and B of S satisfying the given conditions.

Therefore all integers n satisfying the given conditions are even integers with $n \geqslant 4$.