POW 2019-17 0.7?

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For $n\in\mathbb{N}$, $x, y>0$ s.t $x^n+y^n=1$, show that $(LHS)=(1-x)(1-y)\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<0.7$

Solution

If $x=0$ or $y=0$, $(LHS)=0<0.7$. Now let's assume $x, y\in(0,1)$

$\boxed{1}$ Term $(1-x)(1-y)$

If we let $t=x^n$, $y^n=1-t$. Then, $(1-x)(1-y)=(1-t^{\frac{1}{n}})(1-(1-t)^{\frac{1}{n}})$

Define $f(t)=\ln(1-t^{\frac{1}{n}})+\ln(1-(1-t)^{\frac{1}{n}})$ for $t\in(0,1)$. Then, f is differentiable on $(0,1)$ and

$$f'(t)=\frac{(-\frac{1}{n})t^{\frac{1}{n}-1}}{1-t^{1/n}}+\frac{\frac{1}{n}(1-t)^{\frac{1}{n}-1}}{1-(1-t)^{1/n}}=\frac{1}{n}\left(-\frac{t^{\frac{1}{n}-1}}{1-t^{\frac{1}{n}}}+\frac{(1-t)^{\frac{1}{n}-1}}{1-(1-t)^{\frac{1}{n}}}\right)$$

$$=\frac{1}{nt(1-t)}\left(-\frac{(1-t)t^{\frac{1}{n}}}{1-t^{\frac{1}{n}}}+\frac{t(1-t)^{\frac{1}{n}}}{1-(1-t)^{\frac{1}{n}}}\right)$$

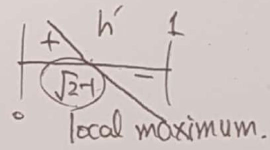
$g(x):=\frac{(1-x)x^{\frac{1}{n}}}{1-x^{\frac{1}{n}}}=\sum_{i=1}^{n}x^{\frac{i}{n}}$ is increasing function on $(0,1)$. Thus,

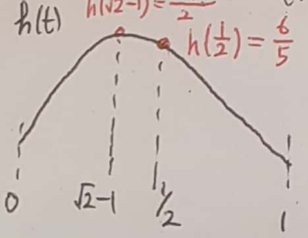
$$f'(t)=\frac{1}{nt(1-t)}\left(-g(t)+g(1-t)\right)\geq 0\iff t\leq 1-t\iff t\leq\frac{1}{2}.$$

$\therefore\ \forall t\in(0,1)$, $f(t)\leq f(\frac{1}{2})=2\ln(1-2^{-\frac{1}{n}})$. and $\underbrace{(1-x)(1-y)=e^{f(x^n)}\leq e^{2\ln(1-2^{-\frac{1}{n}})}}_{}=(1-2^{-\frac{1}{n}})^2$ — ㉮

$\boxed{2}$ Term $\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)=\left(\sum_{k=1}^{n}h(x^{2k})\right)\left(\sum_{k=1}^{n}h(y^{2k})\right)$ for $h(x)=\frac{1+x^2}{1+x^4}$.

Since $h'(t)=\frac{-t^2-2t+1}{(1+t^2)^2}$, $\forall t\in(0,1)$, $h(t)\leq h(\sqrt{2}-1)=\frac{\sqrt{2}+1}{2}$.





The graph of $h(t)$ is as in the left.

Assume $x^n\leq y^n$. Then, $y^n\geq\frac{1}{2}\geq x^n$.

Note that $y^2>y^4>\cdots>y^{2n}$ since $y\in(0,1)$.

Case 1 $2|n$, $n=2k$ for some $k\in\mathbb{N}$.

$y^2>y^4>\cdots>y^{2k}=y^n>\frac{1}{2}$ $\therefore\ \sum_{i=1}^{n}h(y^{2i})=\sum_{i=1}^{k}\underbrace{h(y^{2i})}_{\leq h(\frac{1}{2})}+\sum_{i=k+1}^{n}\underbrace{h(y^{2i})}_{\leq h(\sqrt{2}-1)}\leq\frac{n}{2}\left(\frac{6}{5}+\frac{\sqrt{2}+1}{2}\right)$.

$$\left(\sum_{i=1}^{n}\underbrace{h(x^{2i})}_{\leq h(\sqrt{2}-1)}\right)\left(\sum_{i=1}^{n}h(y^{2i})\right)\leq n\cdot\frac{\sqrt{2}+1}{2}\cdot\frac{n}{2}\left(\frac{6}{5}+\frac{\sqrt{2}+1}{2}\right)=n^2\cdot\frac{\sqrt{2}+1}{4}\left(\frac{6}{5}+\frac{\sqrt{2}+1}{2}\right).$$

<u>Case 2</u> $n=1$. Note that $h(\frac{1}{3}) = h(\frac{1}{2}) = \frac{6}{5}$. Since $x \le \frac{1}{2}$, $x^2 \le \frac{1}{4} < \frac{1}{3}$.

$$\therefore h(x^2)h(y^2) \le h(\tfrac{1}{3})h(\sqrt{2}-1) = \frac{6}{5}\cdot\frac{\sqrt{2}+1}{2} = n^2\cdot\frac{6}{5}\,\frac{\sqrt{2}+1}{2}$$

<u>Case 3</u> $2 \nmid n$, $n>1$. $n = 2k-1$, $k \ge 2$.

$$y^2 > y^4 > \cdots > y^{2k-2} > y^{2k-1} = y^n \ge \frac{1}{2} \quad \therefore \sum_{i=1}^{n} h(y^{2i}) = \sum_{i=1}^{k-1} \underbrace{h(y^{2i})}_{\le h(\frac{1}{2})} + \sum_{i=k}^{n} \underbrace{h(y^{2i})}_{\le h(\sqrt{2}-1)}$$

$$\le (k-1)\cdot\frac{6}{5} + k\cdot\frac{\sqrt{2}+1}{2}.$$

Note that

$$(k-1)\frac{6}{5} + k\frac{\sqrt{2}+1}{2} - (2k-1)\left(\frac{1}{3}\cdot\frac{6}{5} + \frac{2}{3}\cdot\frac{\sqrt{2}+1}{2}\right) = \frac{1}{3}\underbrace{\left(\frac{\sqrt{2}+1}{2} - \frac{6}{5}\right)}_{>0}\underbrace{(2-k)}_{\le 0} \le 0.$$

$$\therefore \sum_{i=1}^{n} h(y^{2i}) \le (2k-1)\left(\frac{1}{3}\cdot\frac{6}{5} + \frac{2}{3}\cdot\frac{\sqrt{2}+1}{2}\right) = n\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right).$$

$$\therefore \left(\sum_{i=1}^{n} \underbrace{h(x^{2i})}_{\le h(\sqrt{2}-1)}\right)\left(\sum_{i=1}^{n} h(y^{2i})\right) \le n\cdot\frac{\sqrt{2}+1}{2}\cdot n\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right) = n^2\cdot\frac{\sqrt{2}+1}{2}\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right).$$

Since $\max\left(\frac{\sqrt{2}+1}{4}\left(\frac{6}{5} + \frac{\sqrt{2}+1}{2}\right), \frac{6}{5}\cdot\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2}\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right)\right) = \frac{\sqrt{2}+1}{2}\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right)$,

from <u>Case 1</u> ~ <u>Case 3</u>, $\forall n \in \mathbb{N}$, $\left(\sum_{k=1}^{n} h(x^{2k})\right)\left(\sum_{k=1}^{n} h(y^{2k})\right) \le \frac{\sqrt{2}+1}{2}\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right) n^2$ — ㉯

From ㉮ and ㉯, $(LHS) \le \frac{\sqrt{2}+1}{2}\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right)\left\{n\left(1 - 2^{-\frac{1}{n}}\right)\right\}^2$.

Since $e^{-s} + s - 1 \ge 0 \ \forall s \ge 0$, $1 - 2^{-\frac{1}{n}} = 1 - e^{-\frac{\ln 2}{n}} \le \frac{\ln 2}{n}$ (put $s = \frac{\ln 2}{n}$)

$$\therefore (LHS) \le \frac{\sqrt{2}+1}{2}\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right)\left\{n\underbrace{\left(1 - 2^{-\frac{1}{n}}\right)}_{\le \frac{\ln 2}{n}}\right\}^2$$

$$\le \frac{\sqrt{2}+1}{2}\left(\frac{2}{5} + \frac{\sqrt{2}+1}{3}\right)(\ln 2)^2 \doteqdot 0.69869\cdots$$

$$< 0.7.$$