Def (SQ-universal) Group G is SQ-universal if every countable group is embedded in some factor group of G.

In this sense, the statement is to prove that G is SQ-universal if there is a subgroup H such that $|G:H| \in \mathbb{N}$ and there is an onto homomorphism from H to F_2, free group of rank 2.

<u>Lemma 1</u> F_2, free group of rank 2, is SQ-universal. (See "Embedding Theorems for Groups" by G. Higman et al. for details)

<u>Lemma 2</u> Every countable group can be embedded in a countably infinite simple group.

<u>Theorem</u> Let G be a group and $H \leq G$ be a subgroup such that $|G:H| \in \mathbb{N}$. Then,
G is SQ-universal $\Longleftrightarrow$ H is SQ-universal

pf) $(\Rightarrow)$ Let K be a countable group. By Lemma 2, $\exists$ countable, infinite and simple group L, $\bar{K} \leq L$ s.t $K \simeq \bar{K} \leq L$. Since G is SQ-universal, $\exists N \trianglelefteq G$ s.t $L \simeq \bar{L} \leq G/N =: \bar{G}$ for some $\bar{L} \leq \bar{G}$.
Let $\bar{H} := HN/N$. Then, $|\bar{G}:\bar{H}| \in \mathbb{N}$ since $|G:H| \in \mathbb{N}$. Thus, $|\bar{L} : \bar{H} \cap \bar{L}| \in \mathbb{N}$.
$\therefore \exists T \leq \bar{H} \cap \bar{L}$ s.t $|\bar{L}:T| \in \mathbb{N}$ and $T \trianglelefteq \bar{L}$. Since $\bar{L}$ is simple, $T = 1$ or $\bar{L}$. Since $\bar{L}$ is infinite, T cannot be trivial ($\because |\bar{L}:T| \in \mathbb{N}$) $\therefore T = \bar{L}$ $\therefore \bar{L} \leq \bar{H} \cap \bar{L} \leq \bar{L}$ $\therefore \bar{H} \cap \bar{L} = \bar{L}$, $\bar{L} \subseteq \bar{H}$.
Then, by 2nd isomorphism theorem, $K \simeq \bar{K} \leq L \simeq \bar{L} \leq \bar{H} = HN/N \simeq H/N \cap H$.
$\therefore$ By definition, H is SQ-universal

$(\Leftarrow)$ Let K be a countable group. By Lemma 2, $\exists$ countable, infinite and simple group L, $\bar{K} \leq L$ s.t $K \simeq \bar{K} \leq L$.
Consider $T = \bigcap_{g \in G} gHg^{-1} \trianglelefteq G$. $|G:T| \in \mathbb{N}$ and $|H:T| \in \mathbb{N}$. ($|G:T| = |G:H| \cdot |H:T|$).
From $(\Rightarrow)$, T is SQ-universal. Thus, there exists $N \trianglelefteq T$, $N \leq R \leq T$ s.t $L \simeq R/N \leq T/N$.

$$\begin{tikzcd} & T/N \arrow[dl, dash] \arrow[dr, dash] & \\ A/N \arrow[dr, dash] & & R/N \arrow[dl, dash] \\ & 1 & \end{tikzcd}$$

We can consider maximal normal subgroup $A \trianglelefteq T$ such that $N \leq A \leq T$ and $A/N \cap R/N$ is trivial. Let $B := RA$. Then,

$$B/A = RA/A \underset{\substack{\uparrow \\ \text{2nd isomorphism theorem}}}{\simeq} R/R \cap A = R/N \simeq L$$

Since L is simple, for group X_0 such that $A \lneq X_0 \trianglelefteq T$, $X \geq B$
(If $A \lneq X_0 \lneq B$, $1 \neq X_0/A \lesssim L$: contradicts the fact that L is simple)
Therefore, every nontrivial normal subgroup of $X := T/A$ contains B/A.
Consider $N_G(A) = \{g \in G \mid gAg^{-1} = A\} \geq T$. $|G : N_G(A)| \leq |G:T|$ is finite. Thus, let $g_1, \cdots, g_n$ be the transversal for $N_G(A)$ and define $A_i := g_i^{-1} A g_i$ $\forall i \in [1,n] \cap \mathbb{N}$.
$\forall i \in [1,n] \cap \mathbb{N}$, $A_i \simeq A \trianglelefteq T \trianglelefteq G$ and $T/A_i \simeq T/A = X$. Let $M := \bigcap_{i=1}^{n} A_i$.
Then, $T/M = T/\bigcap_{i=1}^{n} A_i \simeq T/A_1 \times \cdots \times T/A_n$.
Since $\forall i \in [1,n] \cap \mathbb{N}$, $T/A_i \simeq T/A \geq B/A \simeq L$, T/M has subgroup $\bar{L}$ isomorphic to L
$\therefore K \simeq \bar{K} \leq L \simeq \bar{L} \leq T/M \leq G/M$. Thus, by definition, G is SQ-universal. $\square$

From Lemma 1 and Theorem, given statement is true.