POW 2019-12: Groups Generated by two homeomorphisms of the real line

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Problem 1. *Let I, J be connected open intervals such that $I \cap J$ is a nonempty proper sub-interval of both I and J.*

Let $f, g \in Homeo^+(\mathbb{R})$ such that the set of points of $\mathbb{R}$ which are not fixed by f (g, resp) is precisely I (J, resp.).

Show that for larg enough integer n, the group generated by f^n, g^n is isomorphic to the group with the following presentation

$$\langle a, b \mid [ab^{-1}, a^{-1}ba] = [ab^{-1}, a^{-2}ba^2] = id \rangle .$$

Solution. From now on, let me denote $G = \langle a, b \mid [ab^{-1}, a^{-1}ba] = [ab^{-1}, a^{-2}ba^2] = id \rangle$. This is a group presentation for Thompson group, which is defined using some piecewise linear functions and has the following well-known property.

Lemma 1. *For any $1 \neq N \trianglelefteq G$, N contains all commutators of G.*

It implies that every proper quotient of G is abelian. In addition, replacing a and b by $u = ba^{-1}$ and $v = b^{-1}$, we have an equivalent presentation for G as follows:

Lemma 2.

$$G = \langle u, v \mid [u, vuv(vu)^{-1}] = [u, (vu)^2 v (vu)^{-2}] = 1 \rangle$$

Together with these facts, let me prove the main result. Let $I = (x, z)$ and $J = (y, w)$. we may assume that $x < y < z < w$. Since there is no fixed point of f and g in I and J, respectively, we may assume that $f(y) > y$ and $g(z) > z$ by taking inverse if it is necessary.(Here, note that taking inverse does not affect on the finally generated

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group.) Again from the dynamics of f and g, iteration of g on y converges to w, which implies that

$$z \leq (g^n \circ f)(y) \leq (g^n \circ f^n)(y) \leq (g^{2n} \circ f^n)(y).$$

Now what I want to claim is $\langle f^n, g^n \rangle \cong G$. Note that

$$I \cap (g^n \circ f^n)(J) = (x, z) \cap ((g^n \circ f^n)(y), w) = \emptyset$$

$$I \cap (g^n \circ f^n)^2(J) = (x, z) \cap ((g^{2n} \circ f^n)(y), w) = \emptyset$$

Calculating on each I and J based on the first row of above notification,

$$f^n \circ (g^n \circ f^n \circ g^n \circ (g^n \circ f^n)^{-1}) = (g^n \circ f^n \circ g^n \circ (g^n \circ f^n)^{-1}) \circ f^n.$$

For instance, $f^n \circ (g^n \circ f^n \circ g^n \circ (g^n \circ f^n)^{-1})(I) = f^n \circ (g^n \circ f^n \circ (g^n \circ f^n)^{-1})(I) = f^n(I)$ and $(g^n \circ f^n \circ g^n \circ (g^n \circ f^n)^{-1}) \circ f^n(I) = (g^n \circ f^n \circ (g^n \circ f^n)^{-1}) \circ f^n(I) = f^n(I)$, which are equal. Similarly, second row of above implies that

$$f^n \circ ((g^n \circ f^n)^2 \circ g^n \circ (g^n \circ f^n)^{-2}) = ((g^n \circ f^n)^2 \circ g^n \circ (g^n \circ f^n)^{-2}) \circ f^n.$$

Combining with Lemma 2, we get a well-defined surjective homomorphism

$$\varphi : G \to \langle f^n, g^n \rangle$$

defined by $\varphi(u) = f^n$ and $\varphi(v) = g^n$. So the image $\langle f^n, g^n \rangle$ is isomorphic to some quotient of G. However,

$$(g^n \circ f^n)(J) = ((g^n \circ f^n)(y), w)$$

$$(f^n \circ g^n)(J) = f^n(J) = (f^n(y), f^n(w))$$

so $\langle f^n, g^n \rangle$ is non-abelian. From the Lemma 1, we conclude that φ is actually an isomorphism and

$$G \cong \langle f^n, g^n \rangle .$$

□