POW 2019-13 Property $\mathcal{R}$ 2017XXXX 하석민 ①

Answer: $10 \times 11 = 110$.

pf) Let $\begin{cases} \mathcal{T} = \{A_{a,b} \mid A_{a,b} \text{ has property } \mathcal{R}\} \\ \mathcal{H} = \{A_{a,b} \mid A_{a,b} \in \mathcal{T},\ A_{a-1,b} \notin \mathcal{T},\ A_{a,b-1} \notin \mathcal{T}\} \end{cases}$ Our answer is $\max_{A \in \mathcal{H}} |A|$. Note that $\mathcal{H} \subseteq \mathcal{T}$.

Since $A_{a,b} \in \mathcal{T} \Leftrightarrow A_{b,a} \in \mathcal{T}$, let's only consider $A_{a,b}$ for $a \le b$.

Lemma 1 (Stated this simple fact as Lemma because it is useful)

① $A_{a,b} \in \mathcal{T} \Rightarrow A_{a,x} \in \mathcal{T}\ \ \forall x \ge b$.

② $A_{a,b} \notin \mathcal{T} \Rightarrow A_{a,y} \notin \mathcal{T}\ \ \forall y \le b$

Lemma 2 If $a \le 3$, $A_{a,b} \notin \mathcal{T}$

pf) For coloring $f: A_{a,b} \to \{\text{Red}, \text{Blue}, \text{Yellow}\}$, $f((x,y)) = \begin{cases} \text{Red}, & \text{if } x=1 \\ \text{Blue}, & \text{if } x=2 \\ \text{Yellow}, & \text{if } x=3 \end{cases}$, no row (parallel to x-axis) consists two different point of $A_{a,b}$ with same color, making it impossible for $A_{a,b}$ to have property $\mathcal{R}$. $\therefore A_{a,b} \notin \mathcal{T}$ □.

Our key idea in finding $a, b \in \mathbb{N}$ such that $A_{a,b} \in \mathcal{T}$ is using Pigeonhole Principle (noted PHP from now).
The existance of rectangle (side parallel to axes) with vertices having one color means there are two different rows with same color pair. Note that maximum number of such pair is $\binom{a}{2} \cdot \underset{\substack{\uparrow \\ \text{three color}}}{3}$. Therefore, if (minimum number of same color pair in one row) $\times\, b > \binom{a}{2} \cdot 3$, we can guarantee the existance of such rectangle.

Lemma 3 $A_{4,19} \in \mathcal{T}$.

pf) Fix coloring $f: A_{4,19} \to \overbrace{\{\text{Red}, \text{Blue}, \text{Yellow}\}}^{3 \text{ color}}$. By PHP, $\forall j \in [1,19] \cap \mathbb{N}$, $\exists x_j, y_j \in [1,4] \cap \mathbb{N}$ s.t $x_j \ne y_j$, $f((x_j, j)) = f((y_j, j))$.
Note that $\{x_j, y_j\} \in \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\} = \Omega \rightarrow 6 \text{ element } \left(= \binom{4}{2}\right)$
Let $B = \{(\{x_j, y_j\}, f(x_j, j)) \mid j \in [1,19] \cap \mathbb{N}\}$. Since $|B| \le |\Omega| \times 3 = 18$, by PHP,
$\exists j, k \in [1,19] \cap \mathbb{N}$ s.t $j \ne k$, $(\{x_j, y_j\}, f(x_j, j)) = (\{x_k, y_k\}, f(x_k, k))$.
Then, rectangle whose vertices are $(x_j, j), (y_j, j), (x_k, k), (y_k, k)$ is only colored with $f(x_j, j)$. □

Thm 4 $A_{4,x} \in \mathcal{H} \Leftrightarrow x = 19$.

pf) $A_{4,x} \in \mathcal{T}$ for $x \ge 19$ (Lemma 1). $\therefore A_{4,x} \notin \mathcal{H}$ for $x > 19$.
$A_{4,18} \notin \mathcal{T}$ by the coloring on the right. $\therefore A_{4,x} \notin \mathcal{H}$ for $x < 19$.
Since $A_{3,19} \notin \mathcal{T}$ and $A_{4,18} \notin \mathcal{T}$, $A_{4,19}$ is the only element of $\mathcal{H}$ of form $A_{4,x}$. □

$x \backslash y$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	R	R	R	Y	Y	Y	B	B	B	R	R	R	Y	Y	Y	B	B	B
2	R	Y	Y	R	R	B	B	R	R	B	B	Y	Y	B	B	Y	Y	R
3	Y	R	B	R	B	R	R	B	Y	B	Y	B	B	Y	R	Y	R	Y
4	B	B	R	B	R	R	Y	Y	B	Y	B	B	R	R	Y	R	Y	Y

Similar calculation can be done for larger a.

a	4	5	6	7	8	9	10
t = (minimum number of same-colored pair in a row)	1 xxyz	2 xxyyz	3 xxyyzz	3+1+1=5 xxxyyzz	3+3+1=7 xxxyyyzz	3+3+3=9 xxxyyyzzz	6+3+3=12 xxxxyyyzzz $\binom{4}{2}+\binom{3}{2}+\binom{3}{2}$
$\binom{a}{2}$	6	10	15	21	28	36	45
$\left[\binom{a}{2} \times 3 / t\right] + 1$	18+1=19	16	16	13	13	13	12

$\left[\binom{a}{2} \times 3 / t\right] + 1$: number of row that guarantee the property $\mathcal{R}$.

↳ note that this doesn't mean that smaller number of rows does not satisfy property $\mathcal{R}$.

↳ This means $A_{6,x} \in \mathcal{T}$ for $x \ge 16$, for example.

We don't have to consider when $a \ge 11$ because of Theorem 7 in the back and our assumption $a \le b$.

Def Set $A \subseteq \mathbb{Z}^2$ is "sad" if there is no rectangle whose side is parallel to axes and vertices are element of A.

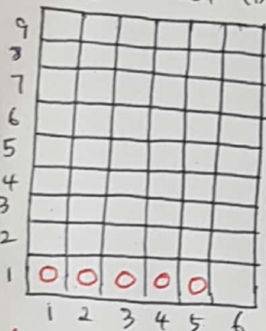
Lemma 5 If $A_{n,m} \notin \gamma$, $\exists$ sad set $A \subseteq A_{n,m}$ s.t $|A| \ge \lceil \frac{nm}{3} \rceil$.

pf) Since $A_{n,m} \notin \gamma$, $\exists$ coloring $f: A_{n,m} \to \{\text{Red, Blue, Yellow}\}$ s.t $\nexists$ rectangle as in definition of property R.
Then, $f^{-1}(\text{Red}), f^{-1}(\text{Blue}), f^{-1}(\text{Yellow})$ is a partition of $A_{n,m}$, thus one of them has at least $\lceil \frac{nm}{3} \rceil$ elements by PHP. □

Lemma 6 If $A \subseteq A_{6,9}$ is sad, $|A| < 22$. If $|A| \ge 22$, proper subset $A' \subseteq A$ s.t $|A'| = 22$ is also sad.

pf) Let $A \subseteq A_{6,9}$ be a sad set. Suppose $|A| = 22$. By PHP, $\exists\ i \in [1,9] \cap \mathbb{N}$ s.t $|\mathbb{R} \times \{i\} \cap A| \ge 3$.
By arranging, let $i=1$ and $(1,1), (1,2), (1,3) \in A$. Then, $\forall j \in [2,9] \cap \mathbb{N}$, $|[1,3] \times \{j\} \cap A| \le 1$
$\therefore |[1,3] \times [2,9] \cap A| \le 9$ and $|[4,6] \times [1,9] \cap A| \ge 22 - 9 = 13$.
By PHP, $\exists\ j \in [1,9]$ s.t $|[4,6] \times \{j\} \cap A| \ge 2$.

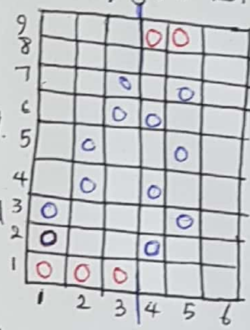
Case 1) $j=1$. Let $(1,4), (1,5) \in A$.



If $(1,6) \in A$, $\forall j \in [2,9] \cap \mathbb{N}$, $|[1,6] \times \{j\} \cap A| \le 1$. $\therefore |A| \le 6 + 8 = 14$ ↯
If $(1,6) \notin A$, $|[1,5] \times [2,9] \cap A| \le 22 - 13 = 9$. By PHP, $\exists\ j \in [2,9] \cap \mathbb{N}$ s.t $|[1,5] \times \{j\} \cap A| \ge 2$. Then, A is not sad since $[1,5] \cap \mathbb{N} \times \{1\} \subseteq A$. ↯

o in (i,j) means $(i,j) \in A$

Case 2) $j \ne 1$. Let $j = 9$ by rearrangement. Let $(4,9), (5,9) \in A$.



$\forall j \in [2,8] \cap \mathbb{N}$, $|[1,3] \times \{j\}| \le 1$ and $|[4,5] \times \{j\}| \le 1$. Also, there's only $3 \times 2 = 6$ such type (as in table on left)
$\therefore |[1,5] \times [2,8] \cap A| \le 2 \times 6 = 12$
$\therefore |\{6\} \times [1,9] \cap A| \ge 22 - 5 - 2 - 12 = 3$
We can't place three o on $\{6\} \times [1,9]$ maintaining A sad ↯.

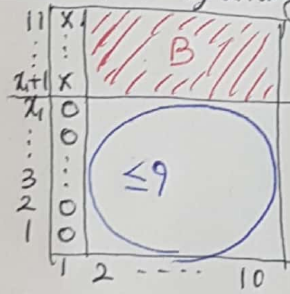
$\therefore$ From Case 1 & 2, A cannot be sad. $\therefore |A| < 22$ □

Theorem 7 $A_{10,11} \in \gamma$.

pf) Let $A \subseteq A_{10,11}$ be a sad set. Define $T_i = \{j \mid (j,i) \in A\}$ and let $x_i = |T_i|$ for $i \in [1,11] \cap \mathbb{N}$. Then, $|A| = \sum_{i=1}^{11} x_i$.
We may assume $x_1 \ge x_2 \ge \cdots \ge x_{11}$. Note that $|T_i \cap T_j| \le 1$ $\forall i,j \in [1,11]$, $i \ne j$ since A is sad.

Case 1 $x_1 \le 3$. $|A| \le 11 x_1 \le 33$. $\therefore |A| \le 33$

Case 2 $x_1 \ge 5$. By arrangement, let $\{1\} \times ([1, x_1] \cap \mathbb{Z}) \subseteq A$.



Since A is sad, $\forall i \in [2,10] \cap \mathbb{N}$, $|T_i \cap [1, x_1]| \le 1$. $\therefore |[2,10] \times [1,x_1] \cap A| \le 9$
Note that $|A|$ is maximized when x_1 is minimized since instead of $(1, x_1) \in A$, if $(1, x_1) \notin A$, we can put at least one element in $([2,10] \times \{x_1\}) \cap \mathbb{Z}^2$ in A maintaining A to be sad.
$\therefore |A| \le 5 + 9 +$ (maximum possible sad subset of 6x9 rectangle B in the left picture)
$x_1 = 5$ $\le 5 + 9 + 21 = 35$ by Lemma 6. $\therefore |A| \le 35$

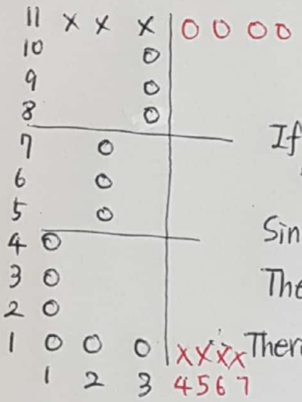
X in (i,j) means $(i,j) \notin A$.

Case 3 $x_1 = 4$

If $x_7 \le 3$, $|A| = \sum_{i=1}^{10} x_i \le \sum_{i=1}^{6} 4 + \sum_{i=7}^{10} 3 = 24 + 12 = 36$ $\therefore |A| \le 36$
Suppose $x_7 = 4$. Then, $x_1 = x_2 = \cdots = x_7 = 4$.
If $\nexists$ distinct number $a, b, c \in [1,7] \cap \mathbb{N}$ s.t $|T_a \cap T_b \cap T_c| = 1$, $\forall j \in [1,11] \cap \mathbb{N}$, $|[1,7] \times \{j\} \cap A| \le 2$.
$\therefore |A \cap [1,7] \times \mathbb{R}| = \sum_{i=1}^{7} x_i = 28 \le 2 \times 11 = 22$ ↯.

Thus, $\exists$ distinct number $a,b,c \in [1,7]\cap\mathbb{N}$ s.t $|T_a\cap T_b\cap T_c| = 1$. By arranging, let $|T_1\cap T_2\cap T_3|=1$

Suppose $T_1\cap T_2\cap T_3=\{1\}$. Assume $\begin{cases} T_1=\{1,2,3,4\} \\ T_2=\{1,5,6,7\} \\ T_3=\{1,8,9,10\} \end{cases}$



If $\nexists$ distinct number $a,b,c,d \in [1,7]\cap\mathbb{N}$ s.t $|T_a\cap T_b\cap T_c\cap T_d|=1$,
$\forall i \in \{4,5,6,7\}$, $1 \notin T_i$.

Since $x_i = |T_i| = 4$ and $|T_1\cap T_i|\le 1$, $|T_2\cap T_i|\le 1$, $|T_3\cap T_i|\le 1$, $11\in T_i$.

Then, $\{11\}\subseteq T_4\cap T_5\cap T_6\cap T_7$: contradiction.

$\therefore$ There exists distinct number $a,b,c,d\in[1,7]\cap\mathbb{N}$ s.t $|T_a\cap T_b\cap T_c\cap T_d|=1$.

By arrangement, let $|T_1\cap T_2\cap T_3\cap T_4|=1$.

$$11 \ge \left|\bigcup_{i=1}^{4} T_i\right| = \sum_{i=1}^{4}|T_i| - \sum_{1\le i<j\le 4}\underbrace{|T_i\cap T_j|}_{\le 1} + \sum_{1\le i<j<k\le 4}\overbrace{|T_i\cap T_j\cap T_k|}^{\ge |T_1\cap T_2\cap T_3\cap T_4|=1} - \underbrace{|T_1\cap T_2\cap T_3\cap T_4|}_{=1}$$

$$\ge 4\times 4 - \binom{4}{2}\times 1 + \binom{4}{3}\times 1 - 1 = 16-6+4-1=13 \quad : \text{Contradiction.}$$

$\therefore$ From Case 1~3, if $A\subseteq A_{10,11}$ is sad, $|A|\le 36$.

Since $\left\lceil \frac{10\times 11}{3}\right\rceil = 37$, by the counterpositive of Lemma 5, $A_{10,11}\in \Upsilon$

$\square$

Lemma 8

$\{A_{10,10}, A_{5,15}, A_{7,12}, A_{9,11}\} \cap \mathcal{Y} = \phi$

pf) ① $A_{5,15} \notin \mathcal{Y}$

$x \backslash y$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	R	R	R	R	B	Y	Y	Y	B	Y	B	Y	B	R	R
2	R	B	Y	B	R	R	R	Y	B	B	Y	R	Y	B	B
3	Y	R	Y	B	R	B	B	R	R	Y	R	B	B	Y	Y
4	B	Y	R	Y	B	R	B	R	Y	R	Y	Y	R	Y	B
5	B	Y	B	R	Y	Y	R	B	R	R	B	B	Y	B	Y

This is a coloring w/o such rectangle.

$\therefore A_{5,15} \notin \mathcal{Y}$

② $A_{7,12}, A_{9,11} \notin \mathcal{Y}$. By Lemma1, enough to show $A_{9,12} \notin \mathcal{Y}$.

	1	2	3	4	5	6	7	8	9
1	R	R	R	B	B	B	Y	Y	Y
2	Y	Y	Y	R	R	R	B	B	B
3	B	B	B	Y	Y	Y	R	R	R
4	R	B	Y	R	B	Y	R	B	Y
5	Y	R	B	Y	R	B	Y	R	B
6	B	Y	R	B	Y	R	B	Y	R
7	R	B	Y	B	Y	R	Y	R	B
8	Y	R	B	R	B	Y	B	Y	R
9	B	Y	R	Y	R	B	R	B	Y
10	R	B	Y	Y	R	B	B	Y	R
11	Y	R	B	B	Y	R	R	B	Y
12	B	Y	R	R	B	Y	Y	R	B

This is a coloring of $A_{9,12}$ w/o such rectangle. $\therefore A_{9,12} \notin \mathcal{Y}$.

$$\left\{\begin{array}{lll} (1\ 2\ 3) & (4\ 5\ 6) & (7\ 8\ 9) \\ (1\ 4\ 7) & (2\ 5\ 8) & (3\ 6\ 9) \\ (1\ 6\ 8) & (2\ 4\ 9) & (3\ 5\ 7) \\ (1\ 5\ 9) & (2\ 6\ 7) & (3\ 4\ 8) \end{array}\right.$$

③ $A_{10,10} \notin \mathcal{Y}$

	1	2	3	4	5	6	7	8	9	10
10	Y	B	R	B	R	R	R	Y	B	Y
9	Y	R	B	R	B	R	Y	R	Y	B
8	R	B	Y	Y	B	B	R	R	Y	Y
7	R	B	B	R	R	Y	Y	B	R	Y
6	Y	R	Y	B	Y	B	R	B	R	B
5	B	B	R	Y	Y	Y	B	R	R	B
4	R	R	R	B	B	Y	B	Y	Y	R
3	B	Y	Y	R	B	Y	R	B	B	R
2	Y	Y	B	Y	R	B	B	R	B	R
1	B	Y	B	B	Y	R	Y	Y	R	R

This is a coloring of $A_{10,10}$ w/o such rectangle. $\therefore A_{10,10} \notin \mathcal{Y}$.

$\square$.

Thm 9

$\mathcal{H} = \{A_{4,19}, A_{19,4}, A_{5,16}, A_{16,5}, A_{7,13}, A_{13,7}, A_{10,11}, A_{11,10}\}$.

pf) By the table in page 1, $A_{4,19}, A_{5,16}, A_{6,16}, A_{7,13}, A_{8,13}, A_{9,13}, A_{10,12} \in \mathcal{Y}$.

- From Thm 4, $A_{4,19} \in \mathcal{H}$.
- From Lemma 8 and proof of Thm 4, $A_{4,16}, A_{5,15} \notin \mathcal{Y}$. $\therefore A_{5,16} \in \mathcal{H}$
- Since, $A_{5,16} \in \mathcal{Y}$, $A_{6,16} \notin \mathcal{H}$.
  $A_{7,13} \in \mathcal{Y}$, $A_{8,13} \notin \mathcal{H}$
  $A_{8,13} \in \mathcal{Y}$, $A_{9,13} \notin \mathcal{H}$.
- From Lemma 8, $A_{7,12} \notin \mathcal{Y}$ $\therefore A_{7,13} \in \mathcal{H}$.
- From Thm 7, $A_{10,11} \in \mathcal{Y}$. $\therefore A_{10,12} \notin \mathcal{H}$. From Lemma 8, $A_{10,10}, A_{9,11} \notin \mathcal{Y}$ $\therefore A_{10,11} \in \mathcal{H}$. $\square$

$\therefore$ Answer is $\max_{A \in \mathcal{H}} |A| = \max\{76, 80, 91, 110\} = 110$