KAIST Math POW 2019-10

2015**** Jeonghwan, Lee

May 29, 2019

Problem.

Let G be a group. A topology on G is said to be a **group topology** if the map $\mu : G \times G \to G$ defined by $\mu(g, h) := g^{-1}h$ is continuous with respect to this topology where $G \times G$ is equipped with the product topology. A group equipped with a group topology is called a **topological group**. When we have two topologies T_1, T_2 on a set S, we write $T_1 \leq T_2$ if T_2 is finer than T_1, which gives a partial order on the set of all topologies on a given set S. Prove or disprove the following statement. : For a given group G, there exists a *unique* minimal group topology on G (minimal with respect to the partial order we described above).

Solution.

We can answer to this question and it will be different largely depending on whether the definition of *group topology* contains the T_0 separation axiom, since many authors includes the *separation axioms* for the definition of *group topology*, such as T_0 separation axiom or Hausdorff property.

(1) The case that the definition of *group topology* does not contain the T_0 separation axiom. :

Let $\mathcal{T}(G)$ be the collection of all *group topologies* on G and we can give a *partial order* on $\mathcal{T}(G)$ by the *inclusion*. Note that they don't need to satisfy the T_0 property. It is clear that the *trivial topology* $\mathcal{T}_t := \{\emptyset, G\}$ is a group topology on G. Since every topology on G is *finer* than the trivial topology on G, $\mathcal{T}_t$ is not only a *minimal element*, but also the *minimum element* of the partially ordered set $(\mathcal{T}(G), \subseteq)$. Thus, there exists a *unique* minimal group topology on G and explicitly, it is the *trivial topology* on G for this case.

(2) The case that the definition of *group topology* contains the T_0 separation axiom. :

Let $\mathcal{H}(G)$ be the collection of all *group topologies* on G with the T_0 separation axiom and we can also give a *partial order* on $\mathcal{H}(G)$ by the *inclusion*. In this case, the *trivial topology* on G does not belong to $\mathcal{H}(G)$. By the following useful lemma, all group topologies that belong to $\mathcal{H}(G)$ satisfy the *Hausdorff property*.

Lemma 1.

Let G be a group and $\mathcal{T}$ be a group topology on G. Then, T.F.A.E. :

1. $\mathcal{T}$ satisfies the T_0 property.

2. $\mathcal{T}$ satisfies the T_1 property.

3. $\mathcal{T}$ satisfies the T_2 property, i.e., $(G, \mathcal{T})$ is Hausdorff.

4. $\mathcal{T}$ satisfies the T_3 property, i.e., $(G, \mathcal{T})$ is regular.

5. $\mathcal{T}$ satisfies the $T_{3\frac{1}{2}}$ property, i.e., $(G, \mathcal{T})$ is completely regular.

Now, let's construct a group that does not satisfy the main statement of the problem. Let's recall the definition of *projective linear group* and *projective special linear group*.

Definition 1.

Let $\mathbb{F}$ be the underlying field.

1. Let $Z_n(\mathbb{F}) := \{tI_n : t \in \mathbb{F}^\times\}$ be the *center* of the *general linear group* $GL_n(\mathbb{F})$. Then, $Z_n(\mathbb{F})$ is a closed normal subgroup of $GL_n(\mathbb{F})$. The quotient group *endowed with the quotient topology*, which is induced from the usual topology on $GL_n(\mathbb{F})$, is called the **projective linear group** over $\mathbb{F}$ and it is denoted by $PGL_n(\mathbb{F})$.

2. Let $SZ_n(\mathbb{F}) := \{tI_n : t \in \mathbb{F}, t^n = 1\}$ be the *center* of the *special linear group* $SL_n(\mathbb{F})$. Then, $SZ_n(\mathbb{F})$ is a closed normal subgroup of $SL_n(\mathbb{F})$. The quotient group *endowed with the quotient topology*, which is induced from the usual topology on $SL_n(\mathbb{F})$, is called the **projective special linear group** over $\mathbb{F}$ and it is denoted by $PSL_n(\mathbb{F})$.

Note that both the real number field $\mathbb{R}$ and the p-adic number field $\mathbb{Q}_p$, for some prime number p, are *completions* of $\mathbb{Q}$ and they are induced from different metric topology on $\mathbb{Q}$. Denote $G_1 := PSL_2(\mathbb{R})$ and $G_2 := PSL_2(\mathbb{Q}_p)$ and suppose they are endowed with the quotient topologies induced from the *usual topologies*. We will use the notation that $\mathcal{G}_i$ is the *group topology* on G_i for $i \in \{1, 2\}$.

Let's take a group $\Gamma := PSL_2(\mathbb{Q})$. Define maps $\imath_i : \Gamma \hookrightarrow G_i$, $i \in \{1, 2\}$ by *inclusion maps* and note that Γ can be *embedded densely* into both G_1 and G_2 by the above inclusion maps. Now, let's give 2 topologies T_1 and T_2 on Γ as follows. : Take a topology T_i on Γ be the *pull-back topology* induced from the embedding $\imath_i : \Gamma \hookrightarrow G_i$, *i.e.*,

$$T_i = \imath_i^*(\mathcal{G}_i) := \left\{(\imath_i)^{-1}(U) : U \in \mathcal{G}_i\right\}, i \in \{1, 2\}.$$

It is possible to define the notion of *completeness* and *completion* of a given topological group.

Definition 2.

Let G be a topological group and $\{g_\alpha : \alpha \in A\}$ be a net in G.

1. $\{g_\alpha : \alpha \in A\}$ is a **left Cauchy net** in G if for every open neighborhood U of 1_G in G, there exists $\alpha_0 \in A$ such that
$$g_\alpha^{-1} \cdot g_\beta \in U \text{ for all } \alpha, \beta > \alpha_0.$$
2. $\{g_\alpha : \alpha \in A\}$ is a **right Cauchy net** in G if for every open neighborhood U of 1_G in G, there exists $\alpha_0 \in A$ such that
$$g_\alpha \cdot g_\beta^{-1} \in U \text{ for all } \alpha, \beta > \alpha_0.$$
3. $\{g_\alpha : \alpha \in A\}$ is a **Cauchy net** in G if it is both left Cauchy and right Cauchy in G.

It is not difficult to prove the following basic properties.

Proposition 1.

1. *Let G be a dense subgroup of a topological group H. If $\{g_\alpha : \alpha \in A\}$ is a net in G that converges to some element $h \in H$, then $\{g_\alpha : \alpha \in A\}$ is a Cauchy net in G.*
2. *Let $\varphi : G \to H$ be a continuous group homomorphism, i.e., a topological group homomorphism. If $\{g_\alpha : \alpha \in A\}$ is a (left/right) Cauchy net in G, then $\{\varphi(g_\alpha) : \alpha \in A\}$ is also a (left/right) Cauchy net in H.*

Definition 3.

A topological group G is **(Raĭkov) complete** if every Cauchy net in G converges in G.

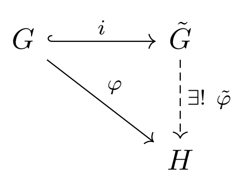
We omit the tedious proof of the next theorems.

Theorem 1 (The existence of Raĭkov completion)**.**

For every Hausdorff topological group G, there exist a complete Hausdorff topological group $\tilde{G}$ and a topological embedding $i : G \hookrightarrow \tilde{G}$ such that $i(G)$ is dense in $\tilde{G}$.

Theorem 2 (The universal property of Raĭkov completion)**.**

If G is a Hausdorff topological group and $\varphi : G \to H$ is a topological group homomorphism, where H is a complete Hausdorff topological group, then there exists a unique topological group homomorphism $\tilde{\varphi} : \tilde{G} \to H$ such that $\varphi = \tilde{\varphi} \circ i$.



From the **Theorem 2**, we can prove the following useful result.

Corollary 1.

Let G be a Hausdorff topological group and $\imath_i : G \hookrightarrow H_i$ be topological embeddings, where H_i are complete Hausdorff topological groups, for $i \in \{1, 2\}$. Then, H_1 and H_2 are topological group isomorphic.

Hence, for any given Hausdorff topological group G, we can define the **(Raĭkov) completion** $(\tilde{G}, i)$ of G, up to *topological group isomorphisms*, as a pair of complete Hausdorff topological group $\tilde{G}$ and a topological embedding $i : G \hookrightarrow \tilde{G}$ *s.t.* $i(G)$ is dense in $\tilde{G}$.

Now, let's return to our main problem. It's obvious that both $(G_1, \mathcal{G}_1)$ and $(G_2, \mathcal{G}_2)$ are *locally compact groups* and therefore they are *complete* from the *well-known fact* that every locally compact group is complete. From the *uniqueness of completion of topological groups*, we can observe that the topological group $(G_i, \mathcal{G}_i)$ is the *completion* of (Γ, T_i) for $i \in \{1, 2\}$. Since the topologies T_1 and T_2 induce different completions of Γ, we get $T_1 \neq T_2$.

We are left to show that the topologies T_1 and T_2 are *minimal* in the partially ordered set of all Hausdorff group topologies on Γ, $\mathcal{H}(\Gamma)$. Let's show that the *usual topology* $\mathcal{G}_i$ is *minimal* on G_i. I want to invoke some awesome results from the paper, *"Equicontinuous Actions of Semisimple Groups"*, written by Uri Bader and Tsachik Gelander.

Theorem 3.

Let G be a semi-simple group and H be any Hausdorff topological group. Also, let $\varphi : G \to H$ be a topological group homomorphism. Then, the image $\varphi(G)$ is closed in H. If further G is separable, then the induced map

$$\tilde{\varphi} : G/\ker(\varphi) \to \varphi(G)$$

is a homeomorphism.

From this theorem, we can prove the following useful property.

Corollary 2.

Every factor group of a separable semi-simple group is topologically minimal, i.e., the endowed group topology is minimal.

Proof.

Let G be a separable semi-simple group and N be a closed normal subgroup of G. Denote the *quotient topology* on the factor group G/N by T. Let S be a coarser Hausdorff group topology on G/N. By setting $H := (G/N, S)$ and consider the *canonical projection* $\pi : G \twoheadrightarrow H$. Since π is continuous, $\ker(\pi) = N$ and G is separable, the induced map $\tilde{\pi} : (G/N, T) \to H$ is a homeomorphism by the **Theorem 3**. Hence, we obtain $T = S$ and this yields the *topological minimality* of T on the factor group G/N.

∎

Since $\mathbb{Q}$ is dense in both $\mathbb{R}$ and $\mathbb{Q}_p$, $\Sigma := SL_2(\mathbb{Q})$ is a dense subgroup of both $F_1 := SL_2(\mathbb{R})$ and $F_2 := SL_2(\mathbb{Q}_p)$. Thus, both F_1 and F_2 are *separable* topological groups because Σ is countable. Let's review some basic notions in the field of *algebraic geometry.*

Definition 4.

1. An **algebraic group** or a **group variety** is a group G that is an *algebraic variety* such that the maps $m : G \times G \to G$ and $i : G \to G$, given by $m(g, h) := gh$ and $i(g) := g^{-1}$, are *morphisms* of algebraic varieties, *i.e.*, regular maps on algebraic varieties. Here, we give the *Zariski topology* on G as an algebraic variety.

2. An **algebraic subgroup** of an algebraic group G is a closed subgroup under the *Zariski topology* on G.

3. For any algebraic group G, denote by $\mathcal{R}(G)$ the *identity component* of the unique *maximal normal solvable subgroup* of G. Then, $\mathcal{R}(G)$, called the **radical** of G, is the unique maximal normal solvable, connected subgroup of G.

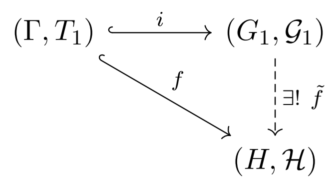
4. An algebraic group G is called **semi-simple** if its radical $\mathcal{R}(G)$ is trivial, *i.e.*, $\mathcal{R}(G) = \{1_G\}$.

It is well-known that for any field $\mathbb{F}$, the *special linear group* over $\mathbb{F}$, $SL_n(\mathbb{F})$ is *semi-simple.* Therefore, both F_1 and F_2 are separable semi-simple groups. From the **Corollary 2**, the factor groups $G_1 = PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/SZ_2(\mathbb{R})$ and $G_2 = PSL_2(\mathbb{Q}_p) = SL_2(\mathbb{Q}_p)/SZ_2(\mathbb{Q}_p)$ are *topologically minimal.* Thus, the usual topology $\mathcal{G}_i$ is *minimal* on G_i.

Claim. *Both T_1 and T_2 are minimal Hausdorff group topologies on Γ.*

Proof of Claim.

Suppose S is any *coarser* Hausdorff group topology on Γ and let $(H, \mathcal{H})$ be the *completion* of (Γ, S) with a topological group embedding $j : (\Gamma, S) \hookrightarrow (H, \mathcal{H})$. Also, let $i : (\Gamma, T_1) \hookrightarrow (G_1, \mathcal{G}_1)$ be a topological group embedding that embeds the group Γ densely into G_1. Define a map $k : (\Gamma, T_1) \to (\Gamma, S)$ which is defined by the *identity map* and it is clearly a *continuous map* since S is coarser than T_1. Let $f := j \circ k : (\Gamma, T_1) \hookrightarrow (H, \mathcal{H})$. By the *universal property of Raĭkov completion*, there exists a unique topological group homomorphism $\tilde{f} : (G_1, \mathcal{G}_1) \to (H, \mathcal{H})$ *s.t.* $f = \tilde{f} \circ i$.



From the definition of *pull-back topology*, note that a map $\varphi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ of topological spaces is continuous if and only if $\varphi^*(\mathcal{T}_Y) := \left\{\varphi^{-1}(U) : U \in \mathcal{T}_Y\right\} \subseteq \mathcal{T}_X$. Since the map $\tilde{f} : (G_1, \mathcal{G}_1) \to (H, \mathcal{H})$ is continuous, we get $\tilde{f}^*(\mathcal{H}) \subseteq \mathcal{G}_1$. The *minimality* of $\mathcal{G}_1$ yields $\tilde{f}^*(\mathcal{H}) = \mathcal{G}_1$. Then, we can deduce that

$$T_1 = i^*(\mathcal{G}_1) = i^*\left\{\tilde{f}^*(\mathcal{H})\right\} = \left(\tilde{f} \circ i\right)^*(\mathcal{H}) = f^*(\mathcal{H}) = j^*(\mathcal{H}) = \left\{j^{-1}\left(j(\Gamma) \cap V\right) : V \in \mathcal{H}\right\} = S,$$

since j is a topological embedding. This implies that T_1 is a *minimal* Hausdorff group topology on Γ and we can show that T_2 is also a minimal Hausdorff group topology on Γ by exactly the same argument.

Hence, the group $\Gamma = PSL_2(\mathbb{Q})$ admits more than one minimal Hausdorff group topologies on Γ ($= T_1$ and T_2). Therefore, there exists a group that admits a minimal Hausdorff group topology, but it is not *unique*. This completes our solution to the given problem. ■