POW 2019-09 2018 Hyunjun Gil

Consider discrete random variable X on the set $\{a_1, a_2 \cdots\}$ with, for some positive integer N,

$$\left\{ \begin{array}{l} a_i = \begin{cases} \frac{\varepsilon}{k}, & i=2k-1, \quad k=1,2,\cdots \\ -\frac{\varepsilon}{k}, & i=2k, \quad k=1,2,\cdots \end{cases} \\ P(X=a_i) = \begin{cases} \frac{1}{2N} & 1 \le i \le 2N \\ 0, & i > 2N \end{cases} \end{array} \right.$$

Then $E(X) = \sum_{n=1}^{\infty} a_n p_n = 0$, $Var(X) = \sum_{n=1}^{\infty} a_n^2 p_n - \left(\sum_{n=1}^{\infty} a_n p_n\right)^2 = \frac{\varepsilon^2}{2N}\left(1^2 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}\right) < \frac{\pi^2 \varepsilon^2}{12N}$

And $H(X) = -\sum_{n=1}^{\infty} p_n \ln p_n = 2N \cdot \frac{1}{2N} \ln 2N = \ln 2N$.

$\therefore e^{2H(X)} \le C_1 \cdot Var(X) + C_2 \Rightarrow 2N \le C_1 \cdot \frac{\varepsilon^2}{2N}\left(1^2 + \cdots + \frac{1}{N^2}\right) + C_2 < C_1 \cdot \frac{\pi^2 \varepsilon^2}{12N} + C_2$

$\therefore$ By setting ε sufficiently small, we can make the term $C_1 \cdot \frac{\varepsilon^2}{2N}\left(1^2 + \cdots + \frac{1}{N^2}\right)$ arbitrary small. Thus, $\forall N \in \mathbb{Z}^+$, $2N \le C_2$

$\therefore$ There are no such constant C_1, C_2.