KAIST Math POW 2019-08

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Problem 8.

Let G be a group acting by isometries on a *proper* geodesic metric space X. Here, X being *proper* means that every closed bounded subset of X is compact. Suppose this action is *proper* and *co-compact*. Here, the action is said to be *proper* if for all compact subsets $K \subseteq X$, the set

$$\{g \in G : g \cdot K \cap K \neq \emptyset\}.$$

is finite. The quotient space X/G is obtained from X by identifying any two points x, y in X if and only if there exists $g \in G$ such that $g \cdot x = y$, and equipped with the *quotient topology*. Then, the action of G on X is said to be *co-compact* if X/G is compact. Under these assumptions, show that G is finitely generated.

Solution.

I want to begin the solution with some basic notions in *geometric group theory*.

Definition 1.

Let (X, d) be a metric space. We say that a group G acts *isometrically* on X if for every $g \in G$, the map $\phi_g : X \to X$ given by $\phi_g(x) := g \cdot x$ is an isometry on X.

Definition 2.

Let (X, d) be a metric space.

1. A path $\gamma : I \to X$, where I is an interval in $\mathbb{R}$, is called a **geodesic** in X if it is an isometric embedding, *i.e.*, $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in I$.

2. (X, d) is called a **geodesic metric space** if for any two points $x, y \in X$, they can be joined by a geodesic in X.

3. For any $r \geq 0$, the **closed r-neighborhood** of a point $x \in X$ is the set

$$\mathcal{N}(x, r) := \{y \in X : d(x, y) \leq r\}.$$

We can also define the r-neighborhood of a subset $Q \subseteq X$ as

$$\mathcal{N}(Q, r) := \bigcup_{x \in Q} \mathcal{N}(x, r).$$

4. A subset Q of X is r**-dense** in X if $X = \mathcal{N}(Q, x)$.

5. A subset Q of X is **co-bounded** in X if it is r-dense in X for some $r \geq 0$.

It is necessary to characterize the *quotient topology* on X/G to analyze the *co-compactness* of the action of G on X more easily!

Proposition 1.

Let (X, d) be a proper metric space and suppose a group G acts isometrically and properly on X. Then, the quotient space X/G is metrizable.

Proof.

Define $d_Q : X/G \times X/G \to [0, \infty)$ by

$$d_Q(G \cdot x, G \cdot y) := \min \{d(p, q) : p \in G \cdot x, q \in G \cdot y\} = \min \{d(x, g \cdot y) : g \in G\}.$$

Note that the second equality comes from the assumption that G acts on X isometrically and d_Q is indeed a metric on G/X. Since G acts properly on X, there are only finitely many $g \in G$ *s.t.* $g \cdot \mathcal{N}(x, d(x, y)) \cap \mathcal{N}(x, d(x, y)) \neq \emptyset$ for any fixed $x, y \in X$. Therefore, there are only finitely many $g \in G$ *s.t.* $d(x, g \cdot y) \leq d(x, y)$ and the minimum $\min \{d(x, g \cdot y) : g \in G\}$ is actually attained! Also, it's straightforward that the metric d_Q on X/G induces the *quotient topology* on X/G. ■

Now, let's return to our main problem. Suppose that (X, d) is a *proper geodesic metric space* and a group G acts *isometrically*, *properly* and *co-compactly* on X. Fix any $x \in X$ and we will show that the orbit $G \cdot x$ is *co-bounded* in X.

Claim 1. *Every orbit of G is co-bounded in X.*

Proof of Claim 1.

Fix any $x \in X$, $\epsilon > 0$ and consider the orbit $G \cdot x \in X/G$. Also, let's consider an open cover $\mathcal{U} := \left\{\mathcal{B}_\epsilon^{d_Q}(G \cdot x) : x \in X\right\}$ of X/G, where $\mathcal{B}_\epsilon^{d_Q}(G \cdot x) := \{G \cdot y \in X/G : d_Q(G \cdot x, G \cdot y) < \epsilon\}$. Since X/G is compact, there exists a finite sub-cover $\left\{\mathcal{B}_\epsilon^{d_Q}(G \cdot x_i) : i \in [n]\right\}$ of $\mathcal{U}$. Let $M := \sup \{d_Q(G \cdot x_i, G \cdot x_j) : i, j \in [n]\} < \infty$ and $r := M + 2\epsilon$. For any $G \cdot y \in X/G$, there exist $i, j \in [n]$ *s.t.* $G \cdot x \in \mathcal{B}_\epsilon^{d_Q}(G \cdot x_i)$ and $G \cdot y \in \mathcal{B}_\epsilon^{d_Q}(G \cdot x_j)$. Then,

$$d_Q(G \cdot x, G \cdot y) \leq d_Q(G \cdot x, G \cdot x_i) + d_Q(G \cdot x_i, G \cdot x_j) + d_Q(G \cdot x_j, G \cdot y) < \epsilon + M + \epsilon = r.$$

Thus, for any $y \in X$, $d_Q(G \cdot x, G \cdot y) = \min \{d(y, g \cdot x) : g \in G\} < r$ and therefore $y \in \mathcal{B}_r^d(g \cdot y) = \{z \in X : d(z, g \cdot y) < r\}$ for some $g \in G$. Hence, $X = \mathcal{N}(G \cdot x, r)$, *i.e.*, the orbit $G \cdot x$ is r-dense in X. This completes the proof of Claim 1.

By the Claim 1, the orbit $G \cdot x$ is r-dense in X for some $r \geq 0$. Let $k := 2r + 1 > 0$ and let us construct a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows. :

- $\mathcal{V} := G$.
- $\mathcal{E} := \left\{ \{g,h\} \in \binom{G}{2} : d(g\cdot x, h\cdot x) \leq k \right\}$.

For any $g,h \in G$, let $L := d(g\cdot x, h\cdot x)$ and $\gamma : [0,L] \to X$ be a geodesic in X with $\gamma(0) = g\cdot x$ and $\gamma(1) = h\cdot x$. Also, we let $n := [L]+1 \in \mathbb{N}$ and note that $\frac{L}{n} < 1$. If we let $x_i := \gamma\left(\frac{L}{n}i\right)$ for $i = 0,1,\cdots,n$, then $d(x_i, x_{i+1}) = d\left(\gamma\left(\frac{L}{n}i\right), \gamma\left(\frac{L}{n}(i+1)\right)\right) = \left|\frac{L}{n}i - \frac{L}{n}(i+1)\right| = \frac{L}{n}$ for all $i = 0,1,\cdots,n-1$. Since the orbit $G\cdot x$ is r-dense in X, for each $i \in [n-1]$, there exists $g_i \in G$ *s.t.* $d(x_i, g_i\cdot x) \leq r$ and set $g_0 := g$ and $g_n := h$. Then, we can observe that for each $i = 0,1,\cdots,n-1$,

$$d(g_i\cdot x, g_{i+1}\cdot x) \leq d(g_i\cdot x, x_i) + d(x_i, x_{i+1}) + d(x_{i+1}, g_{i+1}\cdot x) < r+1+r = k.$$

This yields that $\{g_i, g_{i+1}\} \in \mathcal{E}$ for all $i = 0,1,\cdots,n-1$. Now, it's clear that $g = g_0, g_1, \cdots, g_n = h$ is a path from g to h in $\mathcal{G}$. Therefore, as our intuition (draw a picture and remind the definition of *r-denseness..*), the graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ is connected!

Now, let's prove that the group G is finitely generated. Let $S := \{g \in G : d(x, g\cdot x) \leq k\}$.

Claim 2. *The set S is finite.*

Proof of Claim 2.

Since X is a *proper* metric space, the closed k-neighborhood of x, $\mathcal{N}(x,k)$, is compact. Because the action of G on X is *proper*, the set $T := \{g \in G : g\cdot\mathcal{N}(x,k) \cap \mathcal{N}(x,k) \neq \emptyset\}$ is finite. Since S is a subset of T, the set S is also finite.

Then, we can observe that S is *symmetric*, *i.e.*, for any $g \in G$, $g \in S$ if and only if $g^{-1} \in S$. ($\because$ G acts on X *isometrically.*) To show that the group G is finitely generated, it suffices to prove that the set S generates G.

Claim 3. $G = \langle S \rangle$, *i.e., S generates G.*

Proof of Claim 3.

Note that for any $g,h \in G$, $\{g,h\} \in \mathcal{E}$ iff $gh^{-1} \in S$, since G acts on X *isometrically.* From the *connected-ness* of the graph $\mathcal{G}$, there exists a path in $\mathcal{G}$, $1_G = g_0, g_1, \cdots, g_n = g$ from 1_G to g, where g is any fixed element of G.

$$\therefore\ g = g_n = \left(g_n g_{n-1}^{-1}\right)\cdot\left(g_{n-1}g_{n-2}^{-1}\right)\cdot\ \cdots\ \cdot\left(g_2 g_1^{-1}\right)\left(g_1 g_0^{-1}\right).$$

Since $g_{i+1}g_i^{-1} \in S$ for all $i = 0,1,\cdots,n-1$, $g \in \langle S \rangle$. This yields $G = \langle S \rangle$.

Hence, if (X,d) is a *proper* geodesic metric space and G is a group that acts on X *isometrically*, *properly*, and *co-compactly*, then G is finitely generated. ■

Remark.

The statement of this problem is a *well-known* result in the field of *geometric group theory*, which is called the **Švarc-Milnor Lemma**. Furthermore, if Σ is any finite generating set of G and $x \in X$ is any point, then it can be shown that the *orbit map* $\varphi : (G, d_\Sigma) \to (X, d)$ given by $\varphi(g) := g \cdot x$ is a *quasi-isometry*. Here, d_Σ is the word metric on G corresponding to Σ.