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We begin with the following lemma.

Lemma 1. *If* $g(x) \geqslant 0$ *for all* $x \in \mathbb{R}$ *and* $\int_{-\infty}^{\infty} g(t)dt < \infty$ *then there exist a strictly monotonically increasing sequence* $\{a_n\}_{n\in\mathbb{N}}$ *and a strictly monotonically decreasing sequence* $\{b_n\}_{n\in\mathbb{N}}$ *such that*

$$\left|\lim_{n\to\infty} a_n\right| = \left|\lim_{n\to\infty} b_n\right| = \infty \qquad \textit{and} \qquad \lim_{n\to\infty} g(a_n) = \lim_{n\to\infty} g(b_n) = 0.$$

Proof. By assumption, we must have both $\int_0^{\infty} g(t)dt < \infty$ and $\int_{-\infty}^{0} g(t)dt < \infty$ as g being nonnegative implies both integrals nonnegative. Note that

$$\int_0^{\infty} g(t)dt = \sum_{n=1}^{\infty} \int_{n-1}^{n} g(t)dt < \infty,$$

hence we must have $\lim_{n\to\infty} \int_{n-1}^{n} g(t)dt = 0$. Here, because the length of the interval $[n-1, n]$ is 1, by Mean Value Theorem, for each n there exists $a_n \in (n-1, n)$ such that $\int_{n-1}^{n} g(t)dt = g(a_n)$. Therefore for such a_n's, we have that $\lim_{n\to\infty} g(a_n) = 0$. It is clear from $n-1 < a_n < n$ that $\{a_n\}_{n\in\mathbb{N}}$ is strictly monotonically increasing where its limit is ∞.

On the other hand,

$$\int_{-\infty}^{0} g(t)dt = \sum_{n=1}^{\infty} \int_{-n}^{-n+1} g(t)dt < \infty$$

where the equality is provided from the terms of the sum being positive hence the convergence of the series being absolute. Using the same logic as the previous paragraph, we can find $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} g(b_n) = 0$ and $-n < b_n < -n+1$. In this case $\{b_n\}_{n\in\mathbb{N}}$ is strictly monotonically decreasing where its limit is $-\infty$. □

Now we show the given inequality. Since the maximum of f and the integrals of f^2 and $(f')^2$ from $-\infty$ to ∞ are translation invariant, by translating f we may assume that $f(x)$ attains its maximum at $x = 0$. If $\int_{-\infty}^{\infty} f(t)^2 dt = \infty$ or $\int_{-\infty}^{\infty} (f'(t))^2\, dt = \infty$ then there is nothing to prove, so we may further assume that $\int_{-\infty}^{\infty} f(t)^2 dt < \infty$ and $\int_{-\infty}^{\infty} (f'(t))^2\, dt < \infty$. Then we can choose two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ as in **Lemma 1**,applied to f^2. By AM-GM inequality, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(|f(t)|^2 + |f'(t)|^2\right) dt &\geqslant \int_{-\infty}^{\infty} \left|2f(t)f'(t)\right| dt \\
&= \int_0^{\infty} \left|2f(t)f'(t)\right| dt + \int_{-\infty}^{0} \left|2f(t)f'(t)\right| dt \\
&= \int_0^{\infty} \left|\frac{d}{dt}(f(t))^2\right| dt + \int_{-\infty}^{0} \left|\frac{d}{dt}(f(t))^2\right| dt. \qquad (*)
\end{aligned}$$

Let $V_g(\alpha, \beta)$ denote the total variation of g from α to β. As f^2 is continuous, the total variation of f^2 from α to β is always at least the difference between $f(\alpha)^2$ and $f(\beta)^2$. Here, since $\lim\limits_{n\to\infty} a_n = \infty$ and $\lim\limits_{n\to\infty} b_n = -\infty$, we can write (*) as

$$
\begin{aligned}
&\lim_{n\to\infty} \int_0^{a_n} \left| \frac{d}{dt} (f(t))^2 \right| dt + \lim_{n\to\infty} \int_{b_n}^{0} \left| \frac{d}{dt} (f(t))^2 \right| dt \\
&= \lim_{n\to\infty} V_{f^2}(0, a_n) + \lim_{n\to\infty} V_{f^2}(b_n, 0) \\
&\geqslant \lim_{n\to\infty} \left| f(a_n)^2 - f(0)^2 \right| + \lim_{n\to\infty} \left| f(0)^2 - f(b_n)^2 \right| \\
&= 2f(0)^2 - \lim_{n\to\infty} f(a_n)^2 - \lim_{n\to\infty} f(b_n)^2 \\
&= 2M^2
\end{aligned}
$$

hence the given inequality.