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2017xxxx Keonwoo Kim

There are several available proofs. (In proofs shown below, many integrals are defined improperly or as the Lebesgue integral without explicit notations, and ε and $\tilde{\varepsilon}$ are assumed to be a (small) positive real. Also, **Lemma**s are gathered at the end of this document.)

Proof 1 (Everyone might have tried this). Let

$$I = \int_0^{\pi/2} \log\left(2\cos x\right)\, dx.$$

Note that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, hence substituting with $t = \frac{\pi}{2} - x$, we have

$$I = \int_{\pi/2}^{0} \log\left(2\cos\left(\frac{\pi}{2} - t\right)\right) \cdot (-1)\, dt = \int_0^{\pi/2} \log\left(2\sin t\right)\, dt. \tag{†}$$

Therefore,

$$\begin{aligned}
2I &= I + I \\
&= \int_0^{\pi/2} \log\left(2\cos x\right)\, dx + \int_0^{\pi/2} \log\left(2\sin x\right)\, dx \\
&= \int_0^{\pi/2} \log\left(4\sin x\, \cos x\right)\, dx \\
&= \int_0^{\pi/2} \log\left(2\sin\left(2x\right)\right)\, dx \\
&= \frac{1}{2}\int_0^{\pi} \log\left(2\sin u\right)\, du \qquad\qquad (u = 2x)
\end{aligned}$$

Note that $\sin u$ is symmetric about the vertical line $x = \frac{\pi}{2}$. Hence

$$\frac{1}{2}\int_0^{\pi} \log\left(2\sin u\right)\, du = \int_0^{\pi/2} \log\left(2\sin u\right)\, du = I.$$

Because $2I = I$, we get $I = 0$. □

Proof 2 (With a product of sines). We have (†). By **Lemma 1**,

$$I = \lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{\pi/2} \log(2\sin x)\, dx = \lim_{n\to\infty} \frac{\pi}{n} \sum_{j=1}^{n-1} \log\left(2\sin\left(\frac{j\pi}{n}\right)\right).$$

Also, there is a famous identity (**Lemma 2**):

$$\prod_{j=1}^{n-1} \sin\left(\frac{j\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

Therefore,

$$\begin{aligned} I &= \lim_{n\to\infty} \frac{\pi}{n} \sum_{j=1}^{n-1} \log\left(2\sin\left(\frac{j\pi}{n}\right)\right) \\ &= \lim_{n\to\infty} \frac{\pi}{n} \log\left(2^{n-1} \prod_{j=1}^{n-1} \sin\left(\frac{j\pi}{n}\right)\right) \\ &= \lim_{n\to\infty} \frac{\pi}{n} \log n = 0. \end{aligned}$$

□

Furthermore, there is another way to represent this integral. Note that $\log|\cos x|$ is an antiderivative of $\tan x$, and $\cos x$ is positive on $\left[0, \frac{\pi}{2} - \varepsilon\right]$. Thus,

$$\log(\cos x) = -\int_0^x \tan t\, dt, \qquad \therefore I = \lim_{\varepsilon \searrow 0} \int_0^{\pi/2-\varepsilon} \left(\log 2 - \int_0^x \tan t\, dt\right) dx.$$

By exchanging two integral signs[1],

$$I = \lim_{\varepsilon \searrow 0} \int_0^{\pi/2-\varepsilon} \left(\log 2 - \int_t^{\pi/2-\varepsilon} \tan t\, dx\right) dt = \frac{\pi \log 2}{2} - \lim_{\varepsilon \searrow 0} \int_0^{\pi/2-\varepsilon} \left(\frac{\pi}{2} - t\right) \tan t\, dt.$$

Also, by substituting $x = \frac{\pi}{2} - t$, we have

$$I_1 \coloneqq \lim_{\varepsilon \searrow 0} \int_0^{\pi/2-\varepsilon} \left(\frac{\pi}{2} - t\right) \tan t\, dt = \lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{\pi/2} t \cot t\, dt.$$

So, it suffices to show $I_1 = (\pi \log 2)/2$.

[1] *Tonelli's theorem* works since $\tan t$ is positive; see, for instance, [1, 2.37].

Proof 3 (Using the Leibniz integral rule). Define an integral with a parameter ξ as follows:

$$I(\xi) = \int_0^{\pi/2} f(\xi, x)\, dx \qquad \text{where} \qquad f(\xi, x) = \frac{\tan^{-1}(\xi \tan x)}{\tan x}.$$

Then, we have $I(0) = 0$ and $I(1) = \lim_{\xi \nearrow 1} I(\xi) = I_1$. Using the Leibniz integral rule on $R = \{(x, y) : \varepsilon \le x \le \frac{\pi}{2} - \tilde{\varepsilon},\ 0 \le y \le x\}$, since f and

$$\frac{\partial}{\partial \xi} f(\xi, x) = \frac{1}{(\xi \tan x)^2 + 1}$$

are continuous on a region containing R,

$$\begin{aligned} \frac{\partial}{\partial \xi} \int_\varepsilon^{\pi/2-\tilde{\varepsilon}} f(\xi, x)\, dx &= \int_\varepsilon^{\pi/2-\tilde{\varepsilon}} \frac{\partial}{\partial \xi} f(\xi, x)\, dx \\ &= \int_\varepsilon^{\pi/2-\tilde{\varepsilon}} \frac{1}{(\xi \tan x)^2 + 1}\, dx. \end{aligned}$$

Calculating the integral, $(0 < \xi < 1)$

$$\begin{aligned} \int \frac{1}{(\xi \tan x)^2 + 1}\, dx &= \int \frac{1}{(u^2+1)(\xi^2 u^2 + 1)}\, du && (u = \tan x) \\ &= \frac{1}{\xi^2 - 1}\left(\int \frac{\xi^2}{\xi^2 u^2 + 1}\, du - \int \frac{1}{u^2+1}\, du\right) && \text{(partial fraction)} \\ &= \frac{\xi \tan^{-1}(\xi \tan x) - x}{\xi^2 - 1} + C. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial \xi} \int_\varepsilon^{\pi/2-\tilde{\varepsilon}} f(\xi, x)\, dx &= \left.\frac{\xi \tan^{-1}(\xi \tan x) - x}{\xi^2 - 1}\right|_\varepsilon^{\pi/2-\tilde{\varepsilon}} \\ &= \frac{\xi \tan^{-1}\left(\xi \tan\left(\frac{\pi}{2} - \tilde{\varepsilon}\right)\right) - \xi \tan^{-1}(\xi \tan \varepsilon) - \frac{\pi}{2} + \varepsilon + \tilde{\varepsilon}}{\xi^2 - 1}. \end{aligned}$$

Note that the limit of the above converges uniformly to $(\xi\pi/2 - \pi/2)/(\xi^2 - 1) = \pi/(2(\xi+1))$ as $\varepsilon, \tilde{\varepsilon} \to 0$. Therefore, we can exchange the limit and the differentiation[2]:

$$\frac{\pi}{2(\xi+1)} = \lim_{\varepsilon, \tilde{\varepsilon} \to 0} \frac{\partial}{\partial \xi} \int_\varepsilon^{\pi/2-\tilde{\varepsilon}} f(\xi, x)\, dx = \frac{\partial}{\partial \xi} \lim_{\varepsilon, \tilde{\varepsilon} \to 0} \int_\varepsilon^{\pi/2-\tilde{\varepsilon}} f(\xi, x)\, dx = \frac{\partial}{\partial \xi} I(\xi).$$

Now, by the fundamental theorem of calculus,

$$I(\xi) = I(\xi) - I(0) = \int_0^\xi \frac{\pi}{2(\zeta+1)}\, d\zeta = \frac{\pi}{2} \log(\xi + 1), \qquad \therefore I(1) = \frac{\pi}{2} \log 2. \qquad \square$$

[2]See, e.g., [2, Theorem 7.17].

Proof 4 (Using the Riemann–Lebesgue lemma). By a simple summation of sines (**Lemma 3**), we have the following:

$$\sum_{n=0}^{N} 2\sin(2nx) = \cot x - \frac{\cos((2N+1)x)}{\sin x}.$$

Thus,

$$\int_0^{\pi/2} x\cot x\,dx = 2\sum_{n=0}^{N}\int_0^{\pi/2} x\sin(2x) + \int_0^{\pi/2} x\,\frac{\cos((2N+1)x)}{\sin x}\,dx.$$

Note that

$$\frac{\cos((2N+1)x)}{\sin x} = \cot x\,\cos(2Nx) - \sin(2Nx)$$

whence

$$\int_0^{\pi/2} x\,\frac{\cos((2N+1)x)}{\sin x}\,dx = \int_0^{\pi/2} x\cot x\,\cos(2Nx)\,dx - \int_0^{\pi/2} x\sin(2Nx)\,dx.$$

Here, the RHS converges to 0 as $N \to \infty$ by the Riemann–Lebesgue lemma. Consequently,

$$\int_0^{\pi/2} x\cot x\,dx = 2\sum_{n=0}^{\infty}\int_0^{\pi/2} x\sin(2nx)\,dx = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}\pi}{2n} = \frac{\pi\log 2}{2}. \qquad \square$$

Lemmas

Lemma 1 (convergence of Riemann sums for improper integrals for a monotone function)**.** *Let f be a real-valued monotone function defined on a half-open interval. Suppose, wlog, $f:(0,1]\to\mathbb{R}$ is nonnegative and decreasing. (O/w, consider $C - f(kx)$ for an appropriate constant C and k.) Suppose further that there is a singularity at $x = 0$, but f is Riemann integrable on $[c,1]$ for all $0 < c \le 1$ and the improper integral is convergent:*

$$\int_0^1 f(x)\,dx = \lim_{c\searrow 0}\int_c^1 f(x)\,dx < \infty.$$

Moreover, assume $x\,f(x)\to 0$ as $x \searrow 0$. Then we have:

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx.$$

Proof. Since f is decreasing,

$$\frac{1}{n} f\left(\frac{k}{n}\right) \geq \int_{k/n}^{(k+1)/n} f(x)\,dx \geq \frac{1}{n} f\left(\frac{k+1}{n}\right)$$

so that

$$\frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) \geq \int_{1/n}^{1} f(x)\,dx \geq \frac{1}{n}\sum_{k=2}^{n} f\left(\frac{k}{n}\right),$$

which implies

$$\int_{1/n}^{1} f(x)\,dx + \frac{1}{n} f(1) \leq \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq \int_{1/n}^{1} f(x)\,dx + \frac{1}{n} f\left(\frac{1}{n}\right).$$

Since $f(1)/n$ and $f(1/n)/n$ tend to 0 by the assumptions, by the squeeze theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \lim_{n\to\infty} \int_{1/n}^{1} f(x)\,dx \int_{0}^{1} f(x)\,dx.$$

□

Lemma 2.

$$\prod_{j=1}^{n-1} \sin\left(\frac{j\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

Proof. Consider a root of unity $\zeta = e^{i\,2\pi/n}$. Note that

$$|1-\zeta^k| = \left|1 - \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right| = \sqrt{2 - 2\cos\frac{2k\pi}{n}} = 2\sin\frac{k\pi}{n}.$$

Note that

$$1 + z + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1}(z - \zeta^k).$$

Evaluating the limit as $z \to 1$,

$$n = \prod_{k=1}^{n-1}(1-\zeta^k), \qquad \therefore n = |n| = \prod_{k=1}^{n-1}|1-\zeta^k| = 2^{n-1}\sin\frac{k\pi}{n}.$$

□

Lemma 3.

$$\sum_{n=0}^{N} 2\sin(2nx) = \cot x - \frac{\cos((2N+1)x)}{\sin x}.$$

Proof. When $N = 0$, both sides are the same with 0. Also, an identity

$$2\sin(2Nx) = \frac{\cos((2N-1)x) - \cos((2N+1)x)}{\sin x}$$

completes the induction step:

$$\begin{aligned}
\sum_{n=0}^{N} 2\sin(2nx) &= \sum_{n=0}^{N-1} 2\sin(2nx) + 2\sin(2Nx) \\
&= \left(\cot x - \frac{\cos((2N-1)x)}{\sin x}\right) + \left(\frac{\cos((2N-1)x) - \cos((2N+1)x)}{\sin x}\right) \\
&= \cot x - \frac{\cos((2N+1)x)}{\sin x} \qquad \square
\end{aligned}$$

References

[1] Folland, G. B. 1999. *Real analysis: Modern techniques and their applications.* New York: Wiley.

[2] Rudin, W. 1964. *Principles of mathematical analysis.* McGraw-Hill.