POW 2019-05

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April 7, 2019

**Problem 1.** Let \( p_n \) be the \( n \)-th prime number, i.e. \( p_1 = 2, p_2 = 3, p_3 = 5 \), and so on. Prove that the following series converges:

\[
\sum_{n=1}^{\infty} \frac{1}{p_n} \prod_{k=1}^{n} \frac{p_k - 1}{p_k}. \tag{1}
\]

**Remark.** The condition that the sequence \((p_n)_{n \in \mathbb{N}}\) consists of prime numbers is irrelevant. In fact, the series [1] still converges under the weak assumption \( p_n \geq 1 \). See the proof below.

**Solution.** Let \( f_0 := 1 \). For each \( n \in \mathbb{N} \), let

\[
f_n := \prod_{k=1}^{n} \frac{p_k - 1}{p_k}.
\]

The sequence \((f_0, f_1, f_2, \ldots)\) is positive and decreasing. Since

\[
\sum_{n=1}^{\infty} \frac{1}{p_n} \prod_{k=1}^{n} \frac{p_k - 1}{p_k} = \sum_{n=1}^{\infty} f_n - f_{n-1} \cdot f_n \leq \sum_{n=1}^{\infty} (f_{n-1} - f_n) \leq 1 < \infty,
\]

the series [1] converges. \( \square \)