POW 2019-05

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Problem 1. Let p_n be the n-th prime number, i.e. $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on. Prove that the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \prod_{k=1}^{n} \frac{p_k - 1}{p_k}. \tag{1}$$

Remark. The condition that the sequence $(p_n)_{n\in\mathbf{N}}$ consists of prime numbers is irrelavent. In fact, the series (1) still converges under the weak assumption $p_n \geq 1$. See the proof below.

Solution. Let $f_0 := 1$. For each $n \in \mathbf{N}$, let

$$f_n := \prod_{k=1}^{n} \frac{p_k - 1}{p_k}.$$

The sequence $(f_0, f_1, f_2, \dots)$ is positive and decreasing. Since

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \cdot \prod_{k=1}^{n} \frac{p_k - 1}{p_k} = \sum_{n=1}^{\infty} \frac{f_{n-1} - f_n}{f_{n-1}} \cdot f_n \leq \sum_{n=1}^{\infty} (f_{n-1} - f_n) \leq 1 < \infty,$$

the series (1) converges. □