POW 2019-04

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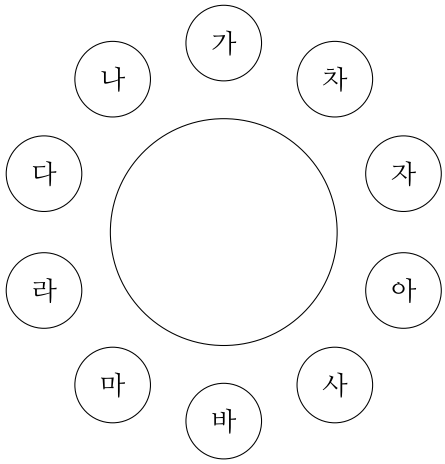
March 32, 2019

Label the mathematicians with 가 to 차 as the following figure.



Let $카_t$ be the amount of food 카 has after t minutes, and define $\mathbf{v}_t = \begin{bmatrix} 가_t & 나_t & \cdots & 차_t \end{bmatrix}^T$.

First consider the case where at each full minute, every mathematician divides his/her share of food into two equal parts and hands it out to the two people seated closest to him in counter-clockwise direction. That is, at each minute, one receives from the two mathematicians closest in clockwise direction, half of the food each had. Then we have $\mathbf{v}_n = \mathbf{A}\mathbf{v}_{n-1}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

and hence $\mathbf{v}_n = \mathbf{A}^n\mathbf{v}_0$. We wish to see what happens as $n \to \infty$.

We attempt to diagonalize $\mathbf{A}$. Let $\zeta = e^{\pi i/5}$. Note that $\zeta^5 = -1$ and $\zeta^{10} = 1$. Observe that, for any $k \in \mathbb{Z}$,

$$\mathbf{A} \begin{bmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{9k} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\zeta^{8k} + \zeta^{9k}\right) \\ \frac{1}{2}\left(\zeta^{9k} + 1\right) \\ \frac{1}{2}\left(1 + \zeta^{k}\right) \\ \vdots \\ \frac{1}{2}\left(\zeta^{7k} + \zeta^{8k}\right) \end{bmatrix} = \frac{1}{2}\left(\zeta^{8k} + \zeta^{9k}\right) \begin{bmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{9k} \end{bmatrix}$$

so $\begin{bmatrix} 1 & \zeta^k & \zeta^{2k} & \cdots & \zeta^{9k} \end{bmatrix}^T$ is an eigenvector with eigenvalue $\frac{1}{2}\left(\zeta^{8k} + \zeta^{9k}\right)$. Thus $\mathbf{A}$ attains a diagonalization

$$\mathbf{A} = \mathbf{S}^{-1}\mathbf{D}\mathbf{S}$$

where

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^9 \\ 1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{18} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^9 & \zeta^{18} & \cdots & \zeta^{81} \end{pmatrix}$$

and

$$\mathbf{D} = \operatorname{diag}\left(\frac{\zeta^0+\zeta^0}{2}, \frac{\zeta^8+\zeta^9}{2}, \frac{\zeta^{16}+\zeta^{18}}{2}, \cdots, \frac{\zeta^{72}+\zeta^{81}}{2}\right).$$

Observe that

$$\mathbf{S}^{-1} = \frac{1}{10}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta^{-1} & \zeta^{-2} & \cdots & \zeta^{-9} \\ 1 & \zeta^{-2} & \zeta^{-4} & \cdots & \zeta^{-18} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{-9} & \zeta^{-18} & \cdots & \zeta^{-81} \end{pmatrix}$$

since indeed the (i,j)- entry of $\mathbf{S}^{-1}\mathbf{S}$ is

$$\frac{1}{10}\sum_{k=0}^{9}\zeta^{-ik}\zeta^{kj} = \frac{1}{10}\sum_{k=0}^{9}\zeta^{(j-i)k} = \begin{cases} \frac{1}{10}\sum_{k=0}^{9} 1 & = 1 \quad \text{if } i = j \\ \frac{1}{10}\frac{1-\zeta^{10(j-i)}}{1-\zeta^{j-i}} & = 0 \quad \text{if } i \neq j. \end{cases}$$

Meanwhile, for any k with $1 \leqslant k \leqslant 9$,

$$\left|\frac{\zeta^{8k}+\zeta^{9k}}{2}\right| = \frac{|\zeta^{8k}|}{2}\cdot\left|1+\zeta^k\right| = \frac{1}{2}\left|1-\zeta^{k+5}\right| < 1$$

because $\left|1-\zeta^{k+5}\right|$ is the length of the chord on a unit circle centered at the origin in the complex plane connecting 1 and $\zeta^{k+5} \neq -1$.

Now that we have $\mathbf{v}_n = \mathbf{A}^n\mathbf{v}_0$ and $\mathbf{A} = \mathbf{S}^{-1}\mathbf{D}\mathbf{S}$, we get

$$\mathbf{v}_n = \mathbf{A}^n\mathbf{v}_0 = \underbrace{\left(\mathbf{S}^{-1}\mathbf{D}\mathbf{S}\right)\left(\mathbf{S}^{-1}\mathbf{D}\mathbf{S}\right)\cdots\left(\mathbf{S}^{-1}\mathbf{D}\mathbf{S}\right)}_{n \text{ times}}\mathbf{v}_0 = \mathbf{S}^{-1}\mathbf{D}^n\mathbf{S}\mathbf{v}_0.$$

Here, as observed above, $\mathbf{D}$ is a diagonal matrix where each diagonal entry except the $(1,1)$-entry has modulus strictly less than 0, and the $(1,1)$-entry is 1, hence

$$\lim_{n\to\infty}\mathbf{D}^n = \operatorname{diag}(1,0,0,\cdots,0).$$

Let $\sigma = 가_0 + 나_0 + \cdots 차_0$. Finally we have

$$\begin{aligned} \lim_{n\to\infty}\mathbf{v}_n &= \mathbf{S}^{-1}\left(\lim_{n\to\infty}\mathbf{D}^n\right)\mathbf{S}\mathbf{v}_0 \\ &= \mathbf{S}^{-1}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta & \cdots & \zeta^9 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^9 & \cdots & \zeta^{81} \end{pmatrix}\mathbf{v}_0 \\ &= \mathbf{S}^{-1}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} 가_0 \\ 나_0 \\ \vdots \\ 차_0 \end{pmatrix} \end{aligned}$$

$$= \frac{1}{10}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta^{-1} & \cdots & \zeta^{-9} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{-9} & \cdots & \zeta^{-81} \end{pmatrix}\begin{pmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma/10 \\ \sigma/10 \\ \vdots \\ \sigma/10 \end{pmatrix},$$

therefore each mathematicians will end up having one tenths of the total food there was at the beginning.

Now consider the case where every mathematician shares his/her food with the two people sitting immediately next to him/her. Then we have $\mathbf{v}_n = \mathbf{A}\mathbf{v}_{n-1}$ where $\mathbf{A}$ is changed to

$$\mathbf{A} = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}.$$

Here

$$\mathbf{A}\begin{bmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{9k} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\zeta^{9k} + \zeta^k\right) \\ \frac{1}{2}\left(1 + \zeta^{2k}\right) \\ \frac{1}{2}\left(\zeta^k + \zeta^{3k}\right) \\ \vdots \\ \frac{1}{2}\left(\zeta^{8k} + 1\right) \end{bmatrix} = \frac{1}{2}\left(\zeta^{-k} + \zeta^k\right)\begin{bmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{9k} \end{bmatrix}$$

so $\begin{bmatrix} 1 & \zeta^k & \zeta^{2k} & \cdots & \zeta^{9k} \end{bmatrix}^T$ is still an eigenvector but the corresponding eigenvalue is

$$\frac{\left(\zeta^k + \zeta^{-k}\right)}{2} = \frac{e^{k\pi i/5} + e^{-k\pi i/5}}{2} = \cos\left(\frac{k\pi}{5}\right).$$

Hence $\mathbf{A}$ attains a diagonalization $\mathbf{A} = \mathbf{S}^{-1}\mathbf{D}\mathbf{S}$ where $\mathbf{S}$ is same as before and

$$\mathbf{D} = \operatorname{diag}\left(\cos 0, \cos\left(\frac{\pi}{5}\right), \cos\left(\frac{2\pi}{5}\right), \cdots, \cos\left(\frac{9\pi}{5}\right)\right).$$

In this case, $\cos\left(\frac{5\pi}{5}\right) = \cos\pi = -1$ and for k such that $1 \leqslant k \leqslant 9$ but $k \neq 5$, $|\cos(k\pi/5)| < 1$. So $\mathbf{D}^n$ does not converge but

$$\begin{aligned} \lim_{n\to\infty} \mathbf{D}^{2n-1} &= \operatorname{diag}(1, 0, 0, 0, 0, -1, 0, 0, 0, 0) \\ \lim_{n\to\infty} \mathbf{D}^{2n} &= \operatorname{diag}(1, 0, 0, 0, 0, 1, 0, 0, 0, 0). \end{aligned}$$

Let $\alpha = 가_0 - 나_0 + 다_0 - 라_0 + \cdots + 자_0 - 차_0$, and $\mathbf{E}_{ij}$ the matrix with (k, l)-entry 1 if $(i, j) = (k, l)$ and 0 otherwise. Then we have

$$\mathbf{S}^{-1} \cdot \mathbf{E}_{66} \cdot \mathbf{S}\mathbf{v}_0$$

$$= \mathbf{S}^{-1} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & \zeta^{25} & \zeta^{30} & \cdots & \zeta^{45} \\ 1 & \cdots & \zeta^{30} & \zeta^{36} & \cdots & \zeta^{54} \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \zeta^{45} & \zeta^{54} & \cdots & \zeta^{81} \end{pmatrix} \mathbf{v}_0$$

$$= \mathbf{S}^{-1} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & -1 & 1 & \cdots & -1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 가_0 \\ \vdots \\ 바_0 \\ 사_0 \\ \vdots \\ 차_0 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & \zeta^{-25} & \zeta^{-30} & \cdots & \zeta^{-45} \\ 1 & \cdots & \zeta^{-30} & \zeta^{-36} & \cdots & \zeta^{-54} \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \zeta^{-45} & \zeta^{-54} & \cdots & \zeta^{-81} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} \alpha \\ -\alpha \\ \vdots \\ \alpha \\ -\alpha \end{pmatrix}$$

where

$$\lim_{n\to\infty} \mathbf{D}^{2n-1} = \mathbf{E}_{11} - \mathbf{E}_{66} \qquad \text{and} \qquad \lim_{n\to\infty} \mathbf{D}^{2n} = \mathbf{E}_{11} + \mathbf{E}_{66}.$$

Using the results until now, we conclude that

$$\begin{aligned} \lim_{n\to\infty} \mathbf{v}_{2n-1} &= \mathbf{S}^{-1} \left(\mathbf{E}_{11} - \mathbf{E}_{66}\right) \mathbf{S}\mathbf{v}_0 \\ &= \mathbf{S}^{-1}\mathbf{E}_{11}\mathbf{S}\mathbf{v}_0 - \mathbf{S}^{-1}\mathbf{E}_{66}\mathbf{S}\mathbf{v}_0 \\ &= \frac{1}{10} \begin{bmatrix} \sigma-\alpha & \sigma+\alpha & \cdots & \sigma-\alpha & \sigma+\alpha \end{bmatrix}^T \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} \mathbf{v}_{2n} &= \mathbf{S}^{-1} \left(\mathbf{E}_{11} + \mathbf{E}_{66}\right) \mathbf{S}\mathbf{v}_0 \\ &= \mathbf{S}^{-1}\mathbf{E}_{11}\mathbf{S}\mathbf{v}_0 + \mathbf{S}^{-1}\mathbf{E}_{66}\mathbf{S}\mathbf{v}_0 \\ &= \frac{1}{10} \begin{bmatrix} \sigma+\alpha & \sigma-\alpha & \cdots & \sigma+\alpha & \sigma-\alpha \end{bmatrix}^T. \end{aligned}$$

where $\sigma + \alpha = 2\,(가_0 + 다_0 + 마_0 + 사_0 + 자_0)$ and $\sigma - \alpha = 2\,(나_0 + 라_0 + 바_0 + 아_0 + 차_0)$.

Therefore in this case we alter between converging to two distinct states.