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Consider

$$\lambda I - T = \begin{pmatrix} \lambda - a_1 & -b_1 & 0 & \cdots & 0 \\ -b_1 & \lambda - a_2 & -b_2 & \ddots & \vdots \\ 0 & -b_2 & \lambda - a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_{N-1} \\ 0 & \cdots & 0 & -b_{N-1} & \lambda - a_N \end{pmatrix}$$

where I is the $N \times N$ identity matrix. Observe that the submatrix obtained by deleting the first row and the last column is an upper triangular matrix with diagonal entries $-b_1, -b_2, \cdots, -b_{N-1}$. Thus the $(1, N)$ minor of $\lambda I - T$ is $(-1)^{N-1} b_1 b_2 \cdots b_{N-1}$, which is nonzero by assumption. Thus $\lambda I - T$ has an $(N-1) \times (N-1)$ submatrix with full rank, hence

$$\operatorname{rank}(\lambda I - T) \geqslant N - 1.$$

By Rank-Nullity Theorem, we must have

$$\operatorname{nullity}(\lambda I - T) \leqslant 1.$$

Therefore , if λ is an eigenvalue then its eigenspace has dimension exactly 1.

Now suppose that T has less than N distinct eigenvalues, say, m eigenvalues where $m < N$. Then by our observation, T can have at most m linearly independent eigenvectors. But since T is a real symmetric matrix, there must exist an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of T, implying that there must be N eigenvectors of T. Therefore our assumption that T has less than N distinct eigenvalues must be false, showing that T must have N distinct eigenvalues.