KAIST POW 2019-01 : Equilateral Polygon

Changki Yun*

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Problem. *Let* Π *be a closed, equilateral* k *- gon with area* A *and perimeter* L*. Prove the following inequality :*

$$\frac{A}{L^2} \le \frac{\cot(\pi/k)}{4k} \le \frac{1}{4\pi}$$

The right side is relatively simple. Since $\cot\theta < 1/\theta$ for positive θ,

$$\frac{\cot(\pi/k)}{4k} < \frac{k}{4k\pi} = \frac{1}{4\pi}. \blacksquare$$

Let's move on to the left side. It can be easily shown that the middle term is the value of A/L^2 when Π is a regular k - gon. So it suffices to prove that A/L^2 is maximized when Π is a regular k - gon. I show the fact that for every non-regular equilateral k-gon Π, there exists an equilateral k-gon Π' with bigger A/L^2.

Case 1. Π *is a triangle.* Obviously, an equilateral triangle is regular. So let's assume that Π has at least 4 vertices.

Case 2. Π *is a concave polygon.*

Let $C(\Pi)$ be a convex hull of Π. As Π is a concave, there is a side e in $C(\Pi)$ which has no intersection with Π except 2 endpoints. Let make a polygon Π' by reflecting sides of Π respect to e, which belongs between 2 endpoints of e.

In sure, Π' is a simple(not self-intersecting), and equilateral k-gon. And it's obvious that Π' has larger area than Π, without gaining its perimeter.$\blacksquare$

Case 3. *All angles of* Π *are less or equal than* π*, but they are not regular.*

From the condition, it can be known that there are 4 vertices P, Q, R, S that PQ, QR, RS are sides of Π, and $\angle PQR \neq \angle QRS$. Let's make a new polygon Π' by 'unifying' the $\angle PQR$ and $\angle QRS$. i.e. by making $\square PQRS$ an isoceles trapezoid. It is trivial that Π' is simple and equilateral. And Π' has bigger area than Π(*).$\blacksquare$

(*) can be shown by the following lemma.

Lemma. *Let* a, b, c, d *be length of 4 sides of a quadrilateral, and* α, γ *be opposite angles, and* K *be the area of the quadrilateral. Then the following relation holds.*

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{\alpha+\gamma}{2}\right).\ (s = \frac{a+b+c+d}{2})$$

*Department of Chemistry, Seoul National University

By Lemma, area of $\square PQRS$ is maximized when $\alpha + \gamma = \pi$, and it is an isoceles quadrilateral since $PQ = RS$. Since area of rest of Π without $\square PQRS$ isn't changed, area of Π' is certainly bigger than area of Π.

Fact. *A convex equilateral polygon with regular angles is regular.*

By applying this fact on the analysis given above, it can be shown that the value of A/L^2 has a maximum when the equilateral polygon is regular. As mentioned, this proves the left side of the inequality. ■