POW 2018-23 Game of polynomials

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November 27, 2018

Answer : Player 2 has the winning strategy.
Note that player 2 can choose three coefficients of $f(x)$. Player 2's winning strategy is to set coefficient of even degree(c_2, c_4, c_6) to be zero as much as possible at his/her first two choice. Right before player 2's last choice, We have $f(x) = g(x) + c_i x^i + c_j x^j$ where $g(x)$ is a monic polynomial of degree 8(with all coefficients determined), and $i, j \in [1, 7] \cap \mathbb{N}, i \neq j (c_i, c_j$ not determined yet). Because of Player 2's strategy, i and j cannot be even at the same time. Thus, there are two cases.

Case 1 : Both i and j are odd

Let $f(x) = g(x) + c_{2a+1}x^{2a+1} + c_{2b+1}x^{2b+1}$ for $2a + 1, 2b + 1 \in [1, 7] \cap \mathbb{N}$. Assume $a > b$. Strategy for Player 2 is to set c_{2a+1} such that $c_{2a+1} > \frac{g(1)+g(-2)2^{-(2b+1)}}{2^{2a-2b}-1}$. Let's show that $f(x)$ has real root regardless of Player 1's choice of c_{2b+1}. Note that

$$f(1) = g(1) + c_{2a+1} + c_{2b+1}$$
$$f(-2) = g(-2) - 2^{2a+1}c_{2a+1} - 2^{2b+1}c_{2b+1}$$

From $c_{2a+1} > \frac{g(1)+g(-2)2^{-(2b+1)}}{2^{2a-2b}-1}$, we have $-g(1) - c_{2a+1} > g(-2)2^{-(2b+1)} - 2^{2a-2b}c_{2a+1}$ and

$$(-\infty, g(-2)2^{-(2b+1)} - 2^{2a-2b}c_{2a+1}) \cap (-g(1) - c_{2a+1}, \infty) = \emptyset.$$

Thus, there is no c_{2b+1} such that $g(-2)2^{-(2b+1)} - 2^{2a-2b}c_{2a+1} - c_{2b+1} = 2^{-(2b+1)}f(-2) > 0$ and $c_{2b+1} - (-g(1) - c_{2a+1}) = f(1) > 0$. This means $f(-2) \leq 0$ or $f(1) \leq 0$ regardless of Player 1's choice of c_{2b+1}. Also, since f is monic, $\lim_{x\to\infty} f(x) = \infty$. Thus, there exists $t > 1$ such that $f(t) > 0$. Since f is continuous, f has real root in interval $[-2, t]$ ($\because [1, t] \subset [-2, t]$) by Intermediate Value Theorem.

Case 2 : One of i, j is odd and one is even

Let $f(x) = g(x) + c_{2a+1}x^{2a+1} + c_{2b}x^{2b}$ for $2a + 1, 2b \in [1, 7] \cap \mathbb{N}$. Strategy for Player 2 is to set c_{2b} such that $c_{2b} < -\frac{g(1)+g(-1)}{2}$. Let's show that $f(x)$ has real root regardless of Player 1's choice of c_{2a+1}. Note that

$$f(1) = g(1) + c_{2a+1} + c_{2b}$$
$$f(-1) = g(-1) - c_{2a+1} + c_{2b}$$

From $c_{2b} < -\frac{g(1)+g(-1)}{2}$, we have $c_{2b} + g(-1) < -c_{2b} - g(1)$ and

$$(-\infty, c_{2b} + g(-1)) \cap (-c_{2b} - g(1), \infty) = \emptyset.$$

Thus, there is no c_{2a+1} such that $c_{2b}+g(-1)-c_{2a+1} = f(-1) > 0$ and $c_{2a+1}-(-c_{2b}-g(1)) = f(1) > 0$. This means $f(-1) \leq 0$ or $f(1) \leq 0$ regardless of Player 1's choice of c_{2a+1}. Also, since f is monic, $\lim_{x\to\infty} f(x) = \infty$. Thus, there exists $t > 1$ such that $f(t) > 0$. Since f is continuous, f has real root in interval $[-1, t]$ ($\because [1, t] \subset [-1, t]$) by Intermediate Value Theorem.

Therefore, Player 2 can always win the game.