POW 2018-22 [redacted] Hyunjun Gil

① (i) ⇒ (ii)

Let for some $c_1, c_2 \in \mathbb{R}^+$, $\forall x \in \mathbb{R}$, $c_1 f_1(x) + c_2 f_2(x) > 0$. Suppose there exist a real x_0 such that $f_1(x_0) \le 0$ and $f_2(x_0) \le 0$. Then $c_1 f_1(x_0) + c_2 f_2(x_0) \le 0 \to \divideontimes$ ∴ There are no such x_0.

∴ (i) ⇒ (ii)

② (ii) ⇒ (i)

'not (i)' is equal to (since (i) is equivalent to $\exists c \in \mathbb{R}^+, \forall x \in \mathbb{R}$, $cf_1(x) + f_2(x) > 0$)

"For any positive real number c, there exist some $x \in \mathbb{R}^+$ such that $cf_1(x) + f_2(x) \le 0$"

⇔ For any positive real number c, the discriminant of $cf_1(x) + f_2(x)$ is non-negative.

So the contraposition of (ii) ⇒ (i) is

"For any positive real number c, the discriminant of $cf_1(x) + f_2(x)$ is non-negative"

⇓

"There exist some x such that $f_1(x) \le 0$ and $f_2(x) \le 0$

If $c \to 0$, the discriminant of f_2, let it D_2, should be non-negative → $D_2 \ge 0$

By symmetricity, the discriminant of f_1, let it D_1, should be non-negative too. → $D_1 \ge 0$

let the range of $f_1(x) \le 0$ and $f_2(x)$ are $\alpha_1 \le x \le \beta_1$, $\alpha_2 \le x \le \beta_2$, respectively.

Suppose there are no such x. Then, WLOG, we can let $\alpha_1 \le \beta_1 < \alpha_2 \le \beta_2$. If then, by lemma, we can find c' such that the discriminant of $c'f_1(x) + f_2(x)$ is negative. ⇒ $\divideontimes$ ∴ There must be such x that $f_1(x) \le 0$ and $f_2(x) \le 0$

∴ (i) ⇔ (ii)

Lemma) If $f_1(x) = (x-\alpha_1)(x-\beta_1)$ and $f_2(x) = (x-\alpha_2)(x-\beta_2)$ where $\alpha_1 \le \beta_1 < \alpha_2 \le \beta_2$ then There exist $c \in \mathbb{R}^+$ such that the discriminant of $cf_1(x) + f_2(x)$ is negative

pf) Let $f_1(x) = x^2 + a_1 x + b_1$ and $f_2(x) = x^2 + a_2 x + b_2$, then since $\beta_1 < \alpha_2$,

$$\frac{-a_1 + \sqrt{a_1^2 - 4b_1}}{2} < \frac{-a_2 - \sqrt{a_2^2 - 4b_2}}{2}$$

$$\Leftrightarrow \quad (a_1 - a_2)^2 > \left(\sqrt{a_1^2 - 4b_1} + \sqrt{a_2^2 - 4b_2}\right)^2$$

$$\Leftrightarrow \quad 2(b_1 + b_2) - a_1 a_2 > \sqrt{(a_1^2 - 4b_1)(a_2^2 - 4b_2)} \ge 0 \quad \cdots ①$$

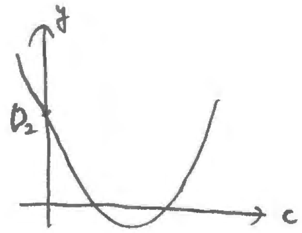
$$\Leftrightarrow \quad (a_1 a_2 - 2(b_1 + b_2))^2 - (a_1^2 - 4b_1)(a_2^2 - 4b_2) > 0 \quad \cdots ②$$

Discriminant of f_1, f_2 and $cf_1 + f_2$ are $D_1 = a_1^2 - 4b_1$, $D_2 = a_2^2 - 4b_2$ and

$$D = (a_1^2 - 4b_1)c^2 + 2(a_1 a_2 - 2(b_1 + b_2))c + (a_2^2 - 4b_2)$$

$$= D_1 c^2 + 2(a_1 a_2 - 2(b_1 + b_2))c + D_2 \quad \cdots ③$$

Since f_1 and f_2 have root, $D_1 \ge 0$ and $D_2 \ge 0$ and by ①, $a_1 a_2 - 2(b_1 + b_2) < 0$. Also by ②, the discriminant of $D(c)$ is positive $\therefore$ the graph of $y = D(c)$ is



$\therefore$ There are some positive real number that makes $D < 0$ ■