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We claim that $C = \frac{1}{2n^2}$ is the only constant that suffices the inequality

$$\frac{C}{a_n} \sum_{1 \leqslant i<j \leqslant n} (a_i - a_j)^2 \leqslant \frac{a_1 + a_2 + \cdots a_n}{n} - \sqrt[n]{a_1 a_2 \cdots a_n} \leqslant \frac{C}{a_1} \sum_{1 \leqslant i<j \leqslant n} (a_i - a_j)^2$$

First we show that the inequality actually holds for $C = \frac{1}{2n^2}$. Observe that

$$\frac{1}{n^2} \sum_{1 \leqslant i<j \leqslant n} (a_i - a_j)^2 = \frac{1}{n^2} \left(n \sum_{i=1}^{n} {a_i}^2 - \left(\sum_{i=1}^{n} a_i \right)^2 \right) = \sum_{i=1}^{n} \frac{a_i^2}{n} - \left(\sum_{i=1}^{n} \frac{a_i}{n} \right)^2 .$$

So we will prove the following stronger inequality; for positive reals $p_1, p_2, \cdots, p_n$ satisfying $\sum_{i=1}^{n} p_i = 1$,

$$\frac{1}{2a_n} \left(\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \right) \leqslant \sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} {a_i}^{p_i} \leqslant \frac{1}{2a_1} \left(\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \right) \quad (*)$$

where the inequality given from the problem is the case where $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$. We use induction on n. The inequality (*) is clear for $n = 1$, because all terms become 0 in this case. So let $n \geqslant 2$. Consider the left inequality of (*). Fix $p_1, p_2, \cdots, p_n$ and $a_1, a_2, \cdots, a_{n-1}$. Varying a_n over the interval $[a_{n-1}, \infty)$ we will find a_n which minimizes

$$f(a_n) := \sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} {a_i}^{p_i} - \frac{1}{2a_n} \left(\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \right).$$

Differentiating we get

$$\begin{aligned}
\frac{df}{da_n} &= p_n - \frac{p_n}{a_n} \prod_{i=1}^{n} {a_i}^{p_i} - \frac{1}{2a_n^2} \left(a_n \left(2p_n a_n - 2p_n \sum_{i=1}^{n} p_i a_i \right) - \sum_{i=1}^{n} p_i a_i^2 + \left(\sum_{i=1}^{n} p_i a_i \right)^2 \right) \\
&= p_n - \frac{p_n}{a_n} \prod_{i=1}^{n} {a_i}^{p_i} - p_n + \frac{p_n}{a_n} \sum_{i=1}^{n} p_i a_i + \frac{1}{2a_n^2} \left(\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \right) \\
&= \frac{p_n}{a_n} \left(\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} {a_i}^{p_i} \right) + \frac{1}{2a_n^2} \left(\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \right).
\end{aligned}$$

The first term is positive by the weighted arithmetic-geometric mean inequality, and the second term is positive as it has the form of a variance of a random variable with a distribution of having a_i as an outcome with probability p_i. Hence $\frac{df}{da_n} \geqslant 0$, and f is increasing. Therefore f is minimized when $a_n = a_{n-1}$. Here, letting $a_n = a_{n-1}$, the left inequality of (*) reduces to the case where there are $n - 1$ variables, a_1 to a_{n-1}, with each in correspondence with p_1, p_2, $\cdots$, p_{n-2}, and $p_{n-1} + p_n$. By induction hypothesis, the minimum of f is nonnegative, which proves the left inequality of (*).

For the right inequality of (*), fix $p_1, p_2, \cdots, p_n$ and $a_2, \cdots, a_{n-1}, a_n$ and consider

$$g(a_1) := \sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} {a_i}^{p_i} - \frac{1}{2a_1}\left(\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2\right)$$

varying a_1 in the interval $(0, a_2]$. Similar calculation and logic can be applied to show that

$$\frac{dg}{da_1} = \frac{p_n}{a_1}\left(\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} {a_i}^{p_i}\right) + \frac{1}{2a_1^2}\left(\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2\right)$$

is also nonnegative, so g is maximized when $a_1 = a_2$. We can again reduce the inequality to the case where there are $n-1$ variables, a_2 to a_n, with each in correspondence with $p_1 + p_2$, p_3, $\cdots$, p_n. Then by the induction hypothesis, the maximum of g is nonpositive, which proves the right inequality of (*).

We have shown that (*) holds for $C = \frac{1}{2n^2}$. We show that the bound given by $C = \frac{1}{2n^2}$ is tight. To show that the lower bound is tight, fix $a_1 = x$ for some positive number x and let $a_2 = \cdots = a_{n-1} = x$. Let $a_n = y$ then the left inequality becomes

$$\frac{C}{y}\left(n(n-1)x^2 + ny^2 - ((n-1)x + y)^2\right) \leqslant \frac{n-1}{n}x + y - x^{\frac{n-1}{n}} y^{\frac{1}{n}} \tag{**}$$

which gives us

$$C \leqslant \frac{\frac{n-1}{n}xy + \frac{1}{n}y^2 - x^{\frac{n-1}{n}} y^{\frac{1}{n}}}{(n-1)^2x^2 - 2(n-1)xy + (n-1)y^2}.$$

Taking limit on the right hand side where y tends to x and applying L'Hôpital's rule twice we get $C \leqslant \frac{1}{2n^2}$. On the other hand, for the upper bound, fix $a_n = x$ for some positive number x and let $a_n = a_{n-1} = \cdots = a_2$. Let $a_1 = y$ then the right inequality becomes exactly (**) where the inequality sign is reversed. Also letting y tend to x here we get $C \geqslant \frac{1}{2n^2}$. Therefore we conclude that if C exists then it must be $\frac{1}{2n^2}$, and the bound given by $C = \frac{1}{2n^2}$ is tight. This also shows that C cannot be independent from n.