POW 2018-20 Almost Linear Function

2017[redacted] Seokmin Ha

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The answer is false.

Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Assuming the Axiom of Choice, we can consider the basis $\mathcal{B}$. Fix $\alpha, \beta \in \mathcal{B}$ such that $\alpha \neq \beta$ and consider the function $\varphi : \mathcal{B} \to \mathbb{R}$ satisfying $\varphi(\alpha) = 0, \varphi(\beta) = 1$. We can extend this function to $f : \mathbb{R} \to \mathbb{R}$ as following:
For any $x \in \mathbb{R} \setminus \{0\}$, there exists $n \in \mathbb{N}$, $b_1, b_2, \cdots, b_n \in \mathcal{B}$, $q_1, q_2 \cdots, q_n \in \mathbb{Q} \setminus \{0\}$ such that

$$x = q_1 b_1 + \cdots + q_n b_n \tag{1}$$

Define

$$f(x) := q_1 \varphi(b_1) + q_2 \varphi(b_2) + \cdots + q_n \varphi(b_n).$$

Set $f(0) = 0$. Since the expression (1) is unique(since $\mathcal{B}$ is a basis and q_i's are nonzero), f is well-defined.
For any $x, y \in \mathbb{R}$, there exists $b_1, b_2, \cdots, b_m \in \mathcal{B}$, $q_1, q_2 \cdots, q_m, r_1, r_2, \cdots, r_m \in \mathbb{Q}$ such that

$$\begin{aligned} x &= q_1 b_1 + \cdots + q_m b_m \\ y &= r_1 b_1 + \cdots + r_m b_m \end{aligned}$$

where $\{b_1, \cdots, b_m\}$ is a set of elements in $\mathcal{B}$ used in expressing x or y. Then,

$$\begin{aligned} f(x+y) &= f((q_1 + r_1)b_1 + \cdots + (q_m + r_m)b_m) = (q_1 + r_1)\varphi(b_1) + (q_2 + r_2)\varphi(b_2) + \cdots + (q_m + r_m)\varphi(b_m) \\ &= (q_1\varphi(b_1) + q_2\varphi(b_2) + \cdots + q_m\varphi(b_m)) + (r_1\varphi(b_1) + r_2\varphi(b_2) + \cdots + r_m\varphi(b_m)) \\ &= f(x) + f(y) \end{aligned}$$

Thus, $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Also, $f(nx) = nf(x)$ for all $n \in \mathbb{N}, x \in \mathbb{R}$ by induction.
Suppose there exists $c \in \mathbb{R}$ such that $|f(x) - cx| \leq 1$ for all $x \in \mathbb{R}$. Then, for all $b \in \mathcal{B}$ and $n \in \mathbb{N}$,

$$|f(nb) - c \cdot nb| = |nf(b) - ncb| = n|f(b) - cb| \leq 1$$

Thus,

$$0 \leq |f(b) - cb| \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $f(b) = cb$ for all $b \in \mathcal{B}$. Then, $f(\alpha) = \varphi(\alpha) = 0 = c\alpha$ and $f(\beta) = \varphi(\beta) = 1 = c\beta$. Since $\alpha \in \mathcal{B}, \alpha \neq 0$. Thus, $c = 0$ and $1 = c\beta = 0$. This is a contradiction.

Remark 1. *The given statement is false even if we change all the 1's in the statement into any fixed number $\epsilon > 0$ by same argument.*