Let
\[ f(x) = 1 + \left( \frac{1}{2} \cdot x \right)^2 + \left( \frac{1}{2} \cdot \frac{3}{4} \cdot x^2 \right)^2 + \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot x^3 \right)^2 + \cdots \]

Prove that
\[ (\sin x)f(\sin x)f'(\cos x) + (\cos x)f(\cos x)f'(\sin x) = \frac{2}{\pi \sin x \cos x}. \]

(Sol)
Define two functions (Elliptic integrals)
\[ K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2 \sin^2 t}}, \quad E(x) = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 t} dt, \quad -1 < x < 1. \]

Then we have
\[ K(x) = \int_0^{\pi/2} (1 - x^2 \sin^2 t)^{-\frac{1}{2}} dt = \int_0^{\pi/2} \sum_{n=0}^{\infty} \left( \frac{-1}{n} \right) (-x^2 \sin^2 t)^n dt. \quad (1) \]

Recall that
\[ \left( \frac{-1}{n} \right) = \frac{(-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdots (-\frac{1}{2} - (n - 1))}{n!} \]
\[ = (-1)^n \cdot \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n - 1}{2n} = (-1)^n a_n. \quad (2) \]

Thus
\[ K(x) = \int_0^{\pi/2} \sum_{n=0}^{\infty} a_n x^{2n} \sin^{2n} t dt \]
\[ = \sum_{n=0}^{\infty} \int_0^{\pi/2} a_n x^{2n} \sin^{2n} t dt \]
\[ = \sum_{n=0}^{\infty} a_n x^{2n} \int_0^{\pi/2} \sin^{2n} t dt. \]

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Note that
\[ \int_0^{\pi/2} \sin^{2n} t \, dt = \left( \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \right) \frac{\pi}{2}. \] (7)

Hence
\[ K(x) = \frac{\pi}{2} \sum_{n=0}^{\infty} (a_n x^n)^2 = \frac{\pi}{2} f(x). \] (8)

Next, we need to observe the following two facts:
\[ x \frac{dE}{dx} = E(x) - K(x). \] (9)

\[ E(x) = x(1 - x^2) \frac{dK}{dx} + (1 - x^2)K(x). \] (10)

proof by direct calculation:
\[ x \frac{dE}{dx} = \int_0^{\pi/2} \frac{-x^2 \sin^2 t}{\sqrt{1 - x^2 \sin^2 t}} \, dt \] (11)
\[ = \int_0^{\pi/2} \frac{1 - x^2 \sin^2 t - 1}{\sqrt{1 - x^2 \sin^2 t}} \, dt \] (12)
\[ = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 t} \, dt - \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2 \sin^2 t}} \, dt \] (13)
\[ = E(x) - K(x). \] (14)

In addition,
\[ \frac{dK}{dx} = \int_0^{\pi/2} \frac{x \sin^2 t}{(\sqrt{1 - x^2 \sin^2 t})^3} \, dt. \] (15)

\[ x \frac{dK}{dx} + K(x) = \int_0^{\pi/2} \frac{x^2 \sin^2 t}{(\sqrt{1 - x^2 \sin^2 t})^3} + \frac{1}{\sqrt{1 - x^2 \sin^2 t}} \, dt. \] (16)

\[ (1 - x^2)(x \frac{dK}{dx} + K(x)) = \int_0^{\pi/2} \frac{1 - x^2}{(\sqrt{1 - x^2 \sin^2 t})^3} \, dt. \] (17)

It is not trivial, but easy to verify that
\[ \sqrt{1 - x^2 \sin^2 t} - \frac{1 - x^2}{(\sqrt{1 - x^2 \sin^2 t})^3} = x^2 \frac{d}{dt} \left( \frac{\sin t \cos t}{\sqrt{1 - x^2 \sin^2 t}} \right). \] (18)
Combining (17) and (18), we can show (10).
The final step is to prove the following
\[ K(x)E(y) + K(y)E(x) - K(x)K(y) = \frac{\pi}{2}, \quad y = \sqrt{1 - x^2}. \] (19)

proof by direct calculation:

\[ \text{Put } P(x) = K(x)E(y) + K(y)E(x) - K(x)K(y) \] (20)

\[ \frac{d(K(x)E(y))}{dx} = \frac{dK(x)}{dx}E(y) + K(x)\frac{dE(y)}{dy}\frac{dy}{dx} \] (21)
\[ \frac{d(K(y)E(x))}{dx} = \frac{dK(y)}{dy}\frac{dy}{dx}E(x) + K(y)\frac{dE(x)}{dx} \] (22)
\[ \frac{d(K(x)K(y))}{dx} = \frac{dK(x)}{dx}K(y) + K(x)\frac{dK(y)}{dy}\frac{dy}{dx} \] (23)

By using (9), (10) and (24)

\[ \frac{dy}{dx} = \frac{-x}{y}. \] (24)

We can prove
\[ \frac{dP(x)}{dx} = 0. \]

\(P(x)\) is constant for \(0 < x < 1\). Let this constant be \(\alpha\), then
\[ P(x) = K(x)E(y) + K(y)E(x) - K(x)K(y) = \alpha. \]

(This is called Legendre’s relation.)
Use (9) and (10) to express the above identity in terms of \(K\) only;
\[ P(x) = K(x)\{y(1-y^2)\frac{dK(y)}{dy} + (1-y^2)K(y)\} + K(y)\{x(1-x^2)\frac{dK(x)}{dx} + (1-x^2)K(x)\} - K(x)K(y). \]

By using \(x^2 + y^2 = 1\) to simplify the equation,
\[ P(x) = xy\{xK(x)K'(y) + yK(y)K'(x)\} = \alpha. \]

We can substitute \(x\) (resp. \(y\)) into \(\sin x\) (resp. \(\cos x\)), then
\[ \sin x \cos x\{(\sin x)K(\sin x)K'(\cos x) + (\cos x)K(\cos x)K'(\sin x)\} = \alpha. \] (25)

By (8),
\[ (\frac{\pi}{2})^2 \sin x \cos x\{(\sin x)f(\sin x)f'(\cos x) + (\cos x)f(\cos x)f'(\sin x)\} = \alpha. \] (26)

We are only left to show
\[ \alpha = \frac{\pi}{2}. \]
This can be shown easily. Recall that
\[ P(x) = K(x)E(y) + K(y)E(x) - K(x)K(y) = \alpha. \] (27)
As always, \( y = \sqrt{1 - x^2} \), and we take the limit \( x \searrow 0 \)
Observe that
\[ \lim_{x \searrow 0} K(x) = \lim_{x \searrow 0} E(x) = \frac{\pi}{2}, \lim_{x \searrow 0} K(y) = \infty, \lim_{x \searrow 0} E(y) = 1. \]
Thus we have
\[ \lim_{x \searrow 0} P(x) = \frac{\pi}{2} + \lim_{x \searrow 0} K(y)(E(x) - K(x)) = \alpha. \]
We will show
\[ \lim_{x \searrow 0} K(y)(E(x) - K(x)) = 0. \]
For sufficiently small \( x \approx 0, 1 - x^2 \sin^2 t > \frac{1}{4} \) for all \( 0 < t < \frac{\pi}{2} \)
Therefore,
\[ \left| \int_0^{\pi/2} \frac{-x^2 \sin^2 t}{\sqrt{1 - x^2 \sin^2 t}} dt \right| \leq x^2 \int_0^{\pi/2} 2 \sin^2 t dt = Cx^2. \]
On the other hand, notice that
\[ \int_0^{\pi/2} \frac{1}{\sqrt{1 - y^2 \sin^2 t}} dt = \int_0^{\pi/2} \frac{1}{\sqrt{1 - (1 - x^2) \sin^2 t}} dt = \int_0^{\pi/2} \frac{1}{\sqrt{\cos^2 t + x^2 \sin^2 t}} dt \]
and
\[ \cos^2 t + x^2 \sin^2 t \geq x^2 \] for all \( 0 < t < \frac{\pi}{2} \)
It gives
\[ \left| \int_0^{\pi/2} \frac{1}{\sqrt{1 - y^2 \sin^2 t}} dt \right| \leq \left| \int_0^{\pi/2} \frac{1}{x} dt \right| = \frac{C'}{x}. \]
Thus,
\[ \left| K(y)(E(x) - K(x)) \right| \leq CC'x \rightarrow 0. \]
In other words,
\[ \lim_{x \searrow 0} K(y)(E(x) - K(x)) = 0. \]
The value of the constant \( \alpha \) is determined. \( \alpha = \frac{\pi}{2} \). Substituting \( \alpha = \frac{\pi}{2} \) into (26) ends the proof. □