POW2018-19 SOL

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Let

$$f(x) = 1 + \left(\frac{1}{2} \cdot x\right)^2 + \left(\frac{1}{2} \cdot \frac{3}{4} \cdot x^2\right)^2 + \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot x^3\right)^2 + \cdots$$

Prove that

$$(\sin x) f(\sin x) f'(\cos x) + (\cos x) f(\cos x) f'(\sin x) = \frac{2}{\pi \sin x \cos x}.$$

(Sol)
Define two functions (Elliptic integrals)

$$K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2 \sin^2 t}},\ E(x) = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 t} dt, -1 < x < 1.$$

Then we have

$$K(x) = \int_0^{\pi/2} (1 - x^2 \sin^2 t)^{-\frac{1}{2}} dt = \int_0^{\pi/2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2 \sin^2 t)^n dt. \quad (1)$$

Recall that

$$\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdots (-\frac{1}{2} - (n-1))}{n!} \quad (2)$$

$$= (-1)^n \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} = (-1)^n a_n. \quad (3)$$

Thus

$$K(x) = \int_0^{\pi/2} \sum_{n=0}^{\infty} a_n x^{2n} \sin^{2n} t dt \quad (4)$$

$$= \sum_{n=0}^{\infty} \int_0^{\pi/2} a_n x^{2n} \sin^{2n} t dt \quad (5)$$

$$= \sum_{n=0}^{\infty} a_n x^{2n} \int_0^{\pi/2} \sin^{2n} t dt. \quad (6)$$

Note that

$$\int_0^{\pi/2} \sin^{2n} t dt = (\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}) \frac{\pi}{2}. \tag{7}$$

Hence

$$K(x) = \frac{\pi}{2} \sum_{n=0}^{\infty} (a_n x^n)^2 = \frac{\pi}{2} f(x). \tag{8}$$

Next, we need to observe the following two facts:

$$x \frac{dE}{dx} = E(x) - K(x). \tag{9}$$

$$E(x) = x(1-x^2) \frac{dK}{dx} + (1-x^2) K(x). \tag{10}$$

proof by direct calculation:

$$\begin{aligned} x \frac{dE}{dx} &= \int_0^{\pi/2} \frac{-x^2 \sin^2 t}{\sqrt{1 - x^2 \sin^2 t}} dt & (11) \\ &= \int_0^{\pi/2} \frac{1 - x^2 \sin^2 t - 1}{\sqrt{1 - x^2 \sin^2 t}} dt & (12) \\ &= \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 t} dt - \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2 \sin^2 t}} dt & (13) \\ &= E(x) - K(x). & (14) \end{aligned}$$

In addition,

$$\frac{dK}{dx} = \int_0^{\pi/2} \frac{x \sin^2 t}{(\sqrt{1 - x^2 \sin^2 t})^3} dt. \tag{15}$$

$$x \frac{dK}{dx} + K(x) = \int_0^{\pi/2} \frac{x^2 \sin^2 t}{(\sqrt{1 - x^2 \sin^2 t})^3} + \frac{1}{\sqrt{1 - x^2 \sin^2 t}} dt. \tag{16}$$

$$(1-x^2)(x \frac{dK}{dx} + K(x)) = \int_0^{\pi/2} \frac{1 - x^2}{(\sqrt{1 - x^2 \sin^2 t})^3} dt. \tag{17}$$

It is not trivial, but easy to verify that

$$\sqrt{1 - x^2 \sin^2 t} - \frac{1 - x^2}{(\sqrt{1 - x^2 \sin^2 t})^3} = x^2 \frac{d}{dt} \Big(\frac{\sin t \cos t}{\sqrt{1 - x^2 \sin^2 t}} \Big). \tag{18}$$

Combining (17) and (18), we can show (10).
The final step is to prove the following

$$K(x)E(y) + K(y)E(x) - K(x)K(y) = \frac{\pi}{2},\ y = \sqrt{1-x^2}. \tag{19}$$

proof by direct calculation:

$$\text{Put } P(x) = K(x)E(y) + K(y)E(x) - K(x)K(y) \tag{20}$$

$$\frac{d(K(x)E(y))}{dx} = \frac{dK(x)}{dx}E(y) + K(x)\frac{dE(y)}{dy}\frac{dy}{dx} \tag{21}$$

$$\frac{d(K(y)E(x))}{dx} = \frac{dK(y)}{dy}\frac{dy}{dx}E(x) + K(y)\frac{dE(x)}{dx} \tag{22}$$

$$\frac{d(K(x)K(y))}{dx} = \frac{dK(x)}{dx}K(y) + K(x)\frac{dK(y)}{dy}\frac{dy}{dx} \tag{23}$$

By using (9), (10) and

$$\frac{dy}{dx} = -\frac{x}{y}, \tag{24}$$

We can prove

$$\frac{dP(x)}{dx} = 0.$$

$P(x)$ is constant for $0 < x < 1$. Let this constant be α, then

$$P(x) = K(x)E(y) + K(y)E(x) - K(x)K(y) = \alpha.$$

(This is called Legendre's relation.)
Use (9) and (10) to express the above identity in terms of K only;

$$P(x) = K(x)\{y(1-y^2)\frac{dK(y)}{dy} + (1-y^2)K(y)\} + K(y)\{x(1-x^2)\frac{dK(x)}{dx} + (1-x^2)K(x)\} - K(x)K(y).$$

By using $x^2 + y^2 = 1$ to simplify the equation,

$$P(x) = xy\{xK(x)K'(y) + yK(y)K'(x)\} = \alpha.$$

We can substitute x(resp. y) into $\sin x$(resp. $\cos x$), then

$$\sin x \cos x\{(\sin x)K(\sin x)K'(\cos x) + (\cos x)K(\cos x)K'(\sin x)\} = \alpha. \tag{25}$$

By (8),

$$\left(\frac{\pi}{2}\right)^2 \sin x \cos x\{(\sin x)f(\sin x)f'(\cos x) + (\cos x)f(\cos x)f'(\sin x)\} = \alpha. \tag{26}$$

We are only left to show

$$\alpha = \frac{\pi}{2}.$$

This can be shown easily.
Recall that

$$P(x) = K(x)E(y) + K(y)E(x) - K(x)K(y) = \alpha. \tag{27}$$

As always, $y = \sqrt{1-x^2}$, and we take the limit $x \searrow 0$
Observe that

$$\lim_{x\searrow 0} K(x) = \lim_{x\searrow 0} E(x) = \frac{\pi}{2}, \lim_{x\searrow 0} K(y) = \infty, \lim_{x\searrow 0} E(y) = 1.$$

Thus we have

$$\lim_{x\searrow 0} P(x) = \frac{\pi}{2} + \lim_{x\searrow 0} K(y)(E(x) - K(x)) = \alpha.$$

We will show

$$\lim_{x\searrow 0} K(y)(E(x) - K(x)) = \lim_{x\searrow 0} \Big(\int_0^{\pi/2} \frac{1}{\sqrt{1-y^2\sin^2 t}}dt\Big)\Big(\int_0^{\pi/2} \frac{-x^2\sin^2 t}{\sqrt{1-x^2\sin^2 t}}dt\Big) = 0.$$

For sufficiently small $x \approx 0, 1 - x^2\sin^2 t > \frac{1}{4}$ for all $0 < t < \frac{\pi}{2}$
Therefore,

$$\Big|\int_0^{\pi/2} \frac{-x^2\sin^2 t}{\sqrt{1-x^2\sin^2 t}}dt\Big| \le x^2 \Big|\int_0^{\pi/2} 2\sin^2 t dt\Big| = Cx^2.$$

On the other hand, notice that

$$\int_0^{\pi/2} \frac{1}{\sqrt{1-y^2\sin^2 t}}dt = \int_0^{\pi/2} \frac{1}{\sqrt{1-(1-x^2)\sin^2 t}}dt = \int_0^{\pi/2} \frac{1}{\sqrt{\cos^2 t + x^2\sin^2 t}}dt$$

and

$$\cos^2 t + x^2\sin^2 t \ge x^2 \text{ for all } 0 < t < \frac{\pi}{2}$$

It gives

$$\Big|\int_0^{\pi/2} \frac{1}{\sqrt{1-y^2\sin^2 t}}dt\Big| \le \Big|\int_0^{\pi/2} \frac{1}{x}dt\Big| = \frac{C'}{x}.$$

Thus,

$$\Big|K(y)(E(x) - K(x))\Big| \le CC'x \longrightarrow 0.$$

In other words,

$$\lim_{x\searrow 0} K(y)(E(x) - K(x)) = 0.$$

The value of the constant α is determined. $\alpha = \frac{\pi}{2}$. Substituting $\alpha = \frac{\pi}{2}$ into (26) ends the proof. □