POW 2018-18 A random walk on the clock

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Let's generalize the problem and consider the case when there are n points evenly spaced on a circle. Denote the points by 1 through n in clockwise order and assume that the particle starts at 1. Denote $q_k = \mathbb{P}(\text{particle P stops at } k \text{ for the last})$ for $k \in \{1, 2, \cdots, n\}$. $q_1 = 0$ since particle starts at 1. Let $p(a, b)$ be the probability that the particle at x visits $x + a$ before visiting $x - b$. This can be considered as a random walk(with equal probability of going left and right) in 1 dimension where a particle starts at the origin and stops when it reaches to a or $-b$. Let p_k be a probability that particle reaches to a before $-b$ starting at $k \in [-b, a] \cap \mathbb{Z}$. Then, $p(a, b) = p_0$. We get the following recurrence relation:

$$p_k = \frac{1}{2}p_{k+1} + \frac{1}{2}p_{k-1}$$

with $p_{-b} = 0, p_a = 1$. The characteristic polynomial of this equation is

$$x = \frac{1}{2}x^2 + \frac{1}{2} \quad \text{or} \quad x^2 - 2x + 1 = 0$$

which has $x = 1$ as double root. Therefore, the general term of p_k can be expressed as

$$p_k = c_1 1^k + c_2 k 1^k = c_1 + c_2 k$$

for some constants c_1 and c_2. Putting the conditions $p_{-b} = 0, p_a = 1$, we have $c_1 = \frac{b}{a+b}$ and $c_2 = \frac{1}{a+b}$. Thus, $p_k = \frac{b+k}{a+b}$ and $p(a, b) = p_0 = \frac{b}{a+b}$.

Fix $k \in \{3, 4, \cdots, n-2\}$. Then, $k-1, k+1 \in \{2, 3, \cdots, n-1\}$. Consider a move of the particle such that k is the last number to be visited. Then, $k-1$ and $k+1$ must be visited before k. Consider the first time when $k-1$ or $k+1$ is visited.

CASE 1 $k-1$ is visited for the first time($k-1$ before $k+1$)
The probability for this case is $p(k-2, n-k)$. After it reaches $k-1$, the probability of k visited last is same as probability of 2 visited last starting from 1, which is same as q_2. Thus, the probability for case 1 is $p(k-2, n-k)q_2 = \frac{n-k}{n-2}q_2$.

CASE 2 $k+1$ is visited for the first time($k+1$ before $k-1$)
The probability for this case is $p(n-k, k-2)$. After it reaches to $k+1$, the probability of k visited last is same as probability of 2 visited last starting from 1(by symmetry), which is same as q_2. Thus, the probability for case 2 is $p(n-k, k-2)q_2 = \frac{k-2}{n-2}q_2$

Thus, $q_k = \frac{n-k}{n-2}q_2 + \frac{k-2}{n-2}q_2 = q_2$ for $k \in \{3, 4, \cdots, n-2\}$. Also, $q_{n-1} = q_2$ by symmetry. Therefore,

$$q_2 = q_3 = \cdots = q_n = \frac{1}{n-1}, q_1 = 0$$

and $(n-1)$ points except for the starting point have the equal probability for the particle P to stop.

Remark 1. *q_2 can also be directly calculated. Starting from 1, in order for 2 be the last visited number, 3 must be visited before 2. Thus,*

$$q_2 = p(n-2, 1) = \frac{1}{n-1}$$