POW 2018-17 Mathematica does not know the answer

[redacted] Seokmin Ha

September 29, 2018

Let $0 < b < a$ be given.

$$\int_0^\infty \frac{e^{ax} - e^{bx}}{x(e^{ax}+1)(e^{bx}+1)} dx = \int_0^\infty \frac{(e^{ax}+1) - (e^{bx}+1)}{x(e^{ax}+1)(e^{bx}+1)} dx = \int_0^\infty \frac{1}{x} \left(\frac{1}{e^{bx}+1} - \frac{1}{e^{bx}+1} \right) dx$$

$$= \int_0^\infty \frac{1}{x} \left[\frac{1}{e^{yx}+1} \right]_{y=a}^{y=b} dx$$

$$= \int_0^\infty \frac{1}{x} \int_a^b \frac{d}{dy} \left(\frac{1}{e^{yx}+1} \right) dy\, dx \quad (\because \text{FTC})$$

$$= \int_0^\infty \frac{1}{x} \int_a^b \left(-\frac{xe^{yx}}{(e^{yx}+1)^2} \right) dy\, dx$$

$$= -\int_0^\infty \int_a^b \frac{e^{yx}}{(e^{yx}+1)^2} dy\, dx$$

$$= -\int_a^b \int_0^\infty \frac{e^{yx}}{(e^{yx}+1)^2} dx\, dy$$

$$= -\int_a^b \int_1^\infty \frac{1}{y(t+1)^2} dt\, dy \qquad (t = e^{yx}, dt = ye^{yx} dx)$$

$$= -\int_a^b \frac{1}{y} \left[-\frac{1}{t+1} \right]_1^\infty dy$$

$$= -\int_a^b \frac{1}{2y} dy$$

$$\therefore \int_0^\infty \frac{e^{ax} - e^{bx}}{x(e^{ax}+1)(e^{bx}+1)} dx = \frac{1}{2} \ln\left(\frac{a}{b}\right)$$