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Now, let's calculate the value.

$$\int_0^\infty \frac{e^{ax}-e^{bx}}{x(e^{ax}+1)(e^{bx}+1)}$$ using

residue theorem.

The root of dominators are

$x=0$, $\frac{(2k+1)\pi i}{a}$, $\frac{(2k+1)\pi i}{b}$.

Now, let $f(x) = \frac{e^{ax}-e^{bx}}{x(e^{ax}+1)(e^{bx}+1)}$

$$\lim_{x \to \frac{(2k+1)\pi i}{b}} f(x)\left(x - \frac{(2k+1)\pi i}{b}\right)$$

$$= -\frac{1}{(2k+1)\pi i} \quad \frac{(2k+1)\pi i}{b} \notin \frac{(2k+1)\pi i}{a}$$

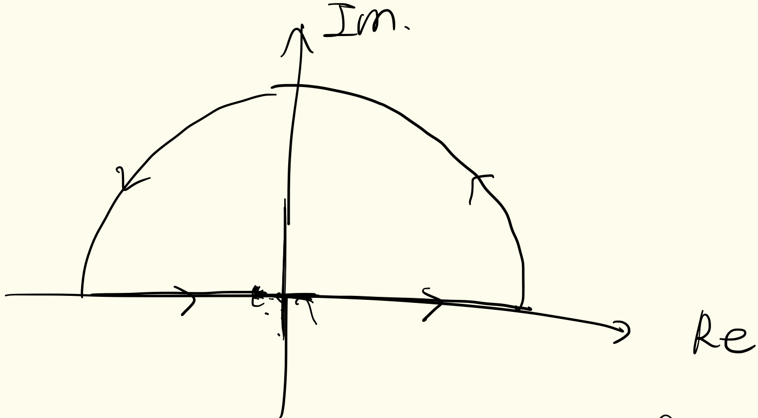
$$\lim_{x \to 0} x f(x) = 0$$

$$\lim_{x \to \frac{(2m+1)\pi i}{a}} f(x)\left(x - \frac{(2m+1)\pi i}{a}\right) = \frac{1}{(2m+1)\pi i}$$

where $\frac{(2m+1)\pi i}{a} \notin \frac{(2\mathbb{Z}+1)\pi i}{b}$

if $\frac{(2m+1)\pi i}{a} = \frac{(2n+1)\pi i}{b} = u$

$$\lim_{x \to u} f(x)(x-u) = \frac{1}{(2m+1)\pi i} - \frac{1}{(2n+1)\pi i}$$



let's consider this half circle with radius R closed path Γ_R and let's apply residue theorem.

Then,

$$\frac{1}{2\pi i}\oint_{F_R} f(z)\,dz = \sum_{\substack{0<\frac{(2m+1)\pi}{a}<R_\ell \\ m\in\mathbb{Z}}} \frac{1}{(2m+1)\pi i} - \sum_{\substack{0<\frac{(2m+1)\pi}{b}<R_\ell \\ m\in\mathbb{Z}}} \frac{1}{(2m+1)\pi i}$$

$$= u(\ell) \quad \text{-----} \quad (*)$$

$$a>b>0 \Rightarrow 0<\frac{1}{a}<\frac{1}{b}$$

Now let's consider

$$C_k = \left[\frac{(2k-1)\pi}{a}, \frac{(2k+1)\pi}{a}\right).$$

$$\bigcup_{k=1}^{\infty} C_k = \left[\frac{\pi}{a}, \infty\right)$$

$$\left|\frac{(2\mathbb{Z}+1)\pi}{b} \cap C_k\right| \le 1.$$

Now let $C_k = D_k \cup E_k$. where.

$$D_k = \left[\frac{(2k-1)\pi}{a}, \frac{2k\pi}{a}\right). \quad E_k = \left[\frac{2k\pi}{a}, \frac{(2k+1)\pi}{a}\right)$$

$\therefore$ There exists infinitely

D_k's s.t. $\left| D_k \cap \frac{(2\mathbb{Z}+1)\pi}{b} \right| = 1$

or E_k's s.t. $\left| E_k \cap \frac{(2\mathbb{Z}+1)\pi}{b} \right| = 1$

W.l.o.g

there exists infinitely

many $D_{i_1}, D_{i_2}, \ldots$ s.t.

$\frac{(2j_\ell+1)\pi}{b} \in D_{i_\ell}$ for some $j_\ell \in \mathbb{N}$.

Then, $\frac{(2\mathbb{Z}+1)\pi}{b} \cap E_{i_\ell} = \emptyset$.

$\therefore$ let $R_\ell = \dfrac{(2\ell+\frac{1}{2})\pi}{a}$,

then $d\left(\dfrac{(2\mathbb{Z}+1)\pi i}{a} \cup \dfrac{(2\mathbb{Z}+1)\pi i}{b}, T_\ell\right)$

$$\geq \frac{\pi}{2a} \quad \text{-----} \quad ①$$

Then we can easily check

that $\exists\, \varepsilon > 0$ & $\exists\, \ell_0 \in \mathbb{N}$ & $\exists\, D > 0$.

s.t. for $\ell > \ell_0$

$\forall z \in A_{\varepsilon,\ell} = \{ |z| = R_\ell \mid Re(z) > 0, Re(z) \leq \varepsilon \}$

$(1+e^{az})(1+e^{bz}) \geq D$ (because ①)

$\therefore$ for $\forall \ell > \ell_0$

$\forall z \in A_{\varepsilon,\ell}$ satisfies $(1+e^{az})(1+e^{bz}) \geq D$

$$\oint_{F_{R_\ell}} f(z)\,dz.$$

$$= \int_{-R_\ell}^{R_\ell} f(z)\,dz + \int_{S_\ell} f(z)\,dz$$

$$\left(\text{where } S_\ell = \{ |z| = R_\ell,\ \operatorname{Re}(z) \geq 0 \}\right)$$

$$= \int_{-R_\ell}^{R_\ell} f(z)\,dz + \int_{A_{\varepsilon,\ell}} f(z)\,dz$$

$$+ \int_{\{|z| = R_\ell,\ \varepsilon < \operatorname{Re}(z) < \sqrt{R_\ell}\}} f(z)\,dz.$$

$$+ \int_{|z| = R_\ell,\ \operatorname{Re}(z) \geq \sqrt{R_\ell}} f(z)\,dz$$

$$\left| \int_{A_{\ell,\varepsilon}} f(z)\,dz \right| \le \frac{2}{DR_\ell} \times \text{length of } A_{\ell,\varepsilon}$$

$$\le \frac{2}{DR_\ell} \times 2 \times 2 \times \frac{\varepsilon}{R_\ell} \times R_\ell = \frac{8\varepsilon}{DR_\ell} = f(\ell)$$

$\left(\frac{\theta}{2} \le \theta - \frac{\theta^3}{3} \le \sin\theta \right.$ $\Rightarrow$ $\theta \le 2\sin\theta$ for sufficiently small θ).

(For sufficiently large ℓ)

$$\left| \int_{|z|=R_\ell,\ \varepsilon < \mathrm{Re}(z) < \sqrt{R_\ell}} f(z)\,dz \right|$$

$$\le \frac{2}{(1-e^{\varepsilon a})(1-e^{\varepsilon b})R_\ell} \times 2 \times 2\sqrt{R_\ell}$$

$$= \frac{8}{(1-e^{\varepsilon a})(1-e^{\varepsilon b})\sqrt{R_\ell}}$$ for sufficiently large ℓ.

$= g(\ell)$

$$\left| \int_{|z|=R_\ell,\ \sqrt{R_\ell} < \mathrm{Re}(z)} f(z)\,dz \right|$$

$$\le 2\pi R_\ell \times \frac{2}{(1+e^{\sqrt{R_\ell}a})(e^{\sqrt{R_\ell}b}-1)} = h(\ell)$$

for sufficiently large ℓ.

$$\lim_{l \to \infty} f(l) + g(l) + h(l) = 0$$

$$\lim_{l \to \infty} \left| \oint_{F_{R_l}} f(z)\,dz - \int_{-R_l}^{R_l} f(z)\,dz \right| = 0$$

Now let's calculate

$$\lim_{l \to \infty} \oint_{F_l} f(z)\,dz = \lim_{l \to \infty} u(l) \times 2\pi i$$

$$(\because (*))$$

$$u(l) = \sum_{\frac{\frac{aR_l}{\pi} - 1}{2} \leq m \leq \frac{\frac{aR_l}{\pi} - 1}{2}} \frac{1}{(2m+1)\pi i}$$

let's prove this

$$\therefore \lim_{x\to\infty} \sum_{ax+b\le y\le cx+d} \frac{1}{y+u} = \ln\frac{c}{a}$$

for $0<a\le c$

pf). $0 \le \frac{1}{u+s} - \int_s^{s+1} \frac{1}{y+u}\,dy \le \frac{1}{u+s} - \frac{1}{u+s+1}$

for $u+s>0$

$$\therefore \sum_{y=k}^{u} \frac{1}{y+u} - \int_k^{u+1} \frac{1}{y+u}\,dy \le \frac{1}{u+k}$$

$\therefore$ For large k,

$$\left| \sum_{ax+b\le y\le cx+d} \frac{1}{y+u} - \int_{ax+b}^{cx+d+1} \frac{1}{y+u}\,dy \right|$$

$$\le \frac{1}{ax+b+u} + \frac{2}{ax+b+u}$$

for sufficiently large x.

$$\lim_{x\to\infty} \frac{2}{ax+b+u} = 0$$

$$\therefore \lim_{x\to\infty} \left| \sum_{ax+b\le y\le cx+d} \frac{1}{y+u} - \int_{ax+b}^{cx+d+1} \frac{1}{y+u}\,dy \right|$$

$$= 0$$

and $$\lim_{x\to\infty} \int_{ax+b}^{cx+d+1} \frac{1}{y+u}\,dy = \lim_{x\to\infty} \ln \frac{cx+d+1}{ax+b+u}$$

$$= \ln \frac{c}{a}$$

$\therefore$ The proposition holds.

$$\therefore \lim_{l\to\infty} u(l) = \frac{1}{2\pi i} \ln \frac{a}{b}$$

$$\therefore \lim_{l\to\infty} \oint_{\Gamma_{R_l}} f(z)\,dz = \ln \frac{a}{b}$$

$$\Rightarrow \lim_{l\to\infty} \int_{-R_l}^{R_l} f(x)\,dx = \ln \frac{a}{b}$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\, dx = \ln \frac{a}{b}$$