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Let $x = (x_1, x_2, \cdots, x_n)$. By symmetry, we may assume that all x_i's are nonnegative, and furthermore $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$.

First assume that $|x_n| > 0$, that is, all entries of x are nonzero. Let

$$B = \left\{ y \in \{-1, 1\}^n \ : \ |y \cdot x| < x_n \right\}.$$

We claim the following.

Claim. $|B| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. If $B = \emptyset$ then there is nothing to prove. Thus suppose $|B| > 0$ and let $B = \{y_1, y_2, \cdots, y_p\}$. For each $y_i = (y_{i1}, y_{i2}, \cdots, y_{in})$, assign a set C_i, a subset of $\{1, 2, \cdots, n\}$, defined as

$$C_i = \left\{ k \in \{1, 2, \cdots, n\} \ : \ y_{ik} = 1 \right\}.$$

Then we can see that $\left\{C_1, C_2, \cdots, C_p\right\}$ is a set of subsets of $\{1, 2, \cdots, n\}$ where none of the elements is a subset of another. To see this, for the sake of contradiction assume that $C_i \subset C_j$ for some $1 \leqslant i, j \leqslant n$, $i \neq j$. Then we have

$$y_j \cdot x - y_i \cdot x = \sum_{m \in C_j \setminus C_i} (x_m - (-x_m)) \geqslant 2 \left(\sum_{m \in C_j \setminus C_i} x_n \right) \geqslant 2x_n$$

which contradicts both $y_j \cdot x$ and $y_i \cdot x$ having absolute value less than x_n.

The Sperner's theorem states that a set of subsets of $\{1, 2, \cdots, n\}$ where none is a subset of another cannot contain more than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements. It follows that

$$|B| = p = \left| \left\{ C_1, C_2, \cdots, C_p \right\} \right| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

and the proof is complete. □

We seek for an upper bound of $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Before we proceed, we first prove the following lemma.

Lemma. $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leqslant \frac{1}{\sqrt{3n+1}}$ *holds for all* $n \in \mathbb{N}$.

Proof. We use induction on n. When $n = 1$ the inequality is clearly true.

Suppose the inequality holds for some $k \in \mathbb{N}$. Observe that the inequality

$$12k^3 + 28k^2 + 19k + 4 \leqslant 12k^3 + 28k^2 + 20k + 4$$

which trivially holds for $k \geqslant 0$ leads to

$$(2k+1)^2(3k+4) \leqslant (2k+2)^2(3k+1) \quad \Longleftrightarrow \quad \frac{2k+1}{2k+2} \leqslant \frac{\sqrt{3k+1}}{\sqrt{3k+4}}$$

therefore we also have

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leqslant \frac{1}{\sqrt{3k+1}} \cdot \frac{\sqrt{3k+1}}{\sqrt{3k+4}} = \frac{1}{\sqrt{3(k+1)+1}}.$$

Hence the proof is complete. □

Using the lemma we have the following theorem.

Theorem. $\binom{n}{\lfloor \frac{n}{2} \rfloor} \leqslant \frac{2^n}{\sqrt{n}}$ *for all* $n \in \mathbb{N}$.

Proof. Suppose n is even, thus $n = 2k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{2k}{k} = \frac{(2k)!}{(k!)^2} &= \frac{2k}{k} \cdot \frac{2k-1}{k} \cdot \frac{2k-2}{k-1} \cdot \frac{2k-3}{k-1} \cdots \frac{3}{2} \cdot \frac{2}{1} \cdot \frac{1}{1} \\ &= 2^{2k} \cdot \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \\ &\leqslant 2^{2k} \cdot \frac{1}{\sqrt{3k+1}} \\ &\leqslant \frac{2^{2k}}{\sqrt{2k}} = \frac{2^n}{\sqrt{n}}.\end{aligned}$$

Now suppose n is odd. We see that the inequality clearly holds when $n = 1$. Thus we may only consider the case when $n = 2k + 1$ for $k \in \mathbb{N}$. In such cases,

$$\begin{aligned}\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{2k+1}{k} = \frac{(2k+1)!}{(k!)(k+1)!} &= \frac{2k+1}{k+1} \cdot \frac{2k}{k} \cdot \frac{2k-1}{k} \cdot \frac{2k-2}{k-1} \cdots \frac{3}{2} \cdot \frac{2}{1} \cdot \frac{1}{1} \\ &= 2^{2k+1} \cdot \frac{2k+1}{2k+2} \cdot \frac{2k-1}{2k} \cdots \frac{3}{4} \cdot \frac{1}{2} \\ &\leqslant 2^{2k+1} \cdot \frac{1}{\sqrt{3k+4}} \\ &\leqslant \frac{2^{2k+1}}{\sqrt{2k+1}} = \frac{2^n}{\sqrt{n}}.\end{aligned}$$

Therefore the given inequality holds for all $n \in \mathbb{N}$. $\square$

With the claim, we obtain the following bound

$$|B| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor} \leqslant \frac{2^n}{\sqrt{n}}.$$

Now assume that we have $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_m > x_{m+1} = \cdots = x_n = 0$, that is, exactly m entries of x are nonzero. Let

$$B_m = \{ y \in \{-1, 1\}^n \ : \ |y \cdot x| < x_m \}.$$

Noting that, if we let $y = (y_1, y_2, \cdots, y_n)$, then we have $y \cdot x = y_1 x_1 + y_2 x_2 + \cdots + y_m x_m$ since $x_{m+1} = \cdots = x_n = 0$. Therefore, for each y_j where $(m+1) \leqslant j \leqslant n$, we can choose either 1 or -1 to be y_j without changing the value of $y \cdot x$. For each y_i where $1 \leqslant i \leqslant m$, as discussed above, there are at most $\frac{2^m}{\sqrt{m}}$ ways to choose the signs of y_i so that $|y \cdot x| < x_m$. Therefore we obatin the following bound

$$|B_m| \leqslant 2^{n-m} \cdot \frac{2^m}{\sqrt{m}} = \frac{2^n}{\sqrt{m}}.$$

Finally we consider the case where x has at least k nonzero entries. This is equivalent to saying that x has m nonzero entries for some $k \leqslant m \leqslant n$. Then for

$$A = \{ y \in \{-1, 1\}^n \ : \ |y \cdot x| = 0 \}$$

it is clear, by definition, that $A \subset B_m$. Therefore we have

$$|A| \leqslant |B_m| \leqslant \frac{2^n}{\sqrt{m}} \leqslant \frac{2^n}{\sqrt{k}}$$

which is exactly the bound desired.