I claim that $m = 1 - \frac{\sqrt{2}}{2}$ is the minimum.

First, let’s prove that $m = 1 - \frac{\sqrt{2}}{2}$ works. Let $f : [0, 1] \rightarrow [0, 1]$ be a convex function. $f$ is continuous on $(0, 1)$, since we may assume that it is also continuous on $0$ and $1$, since we can let $f(0) := \sup_{0 < c < 1/2} \frac{f(1/2) - f(0)}{1/2 - c} (0 - 1/2) + f(1/2)$ and $f(1) := \sup_{0 < c < 1/2} \frac{f(1-c) - f(1/2)}{1-c - 1/2} (1 - 1/2) + f(1/2)$, while the integral doesn’t change. Note that for any $a \in [0, 1]$, $B(a) := \{x \in [0, 1] : f(x) \leq a\}$ is a closed interval if it is nonempty. This is true because $B(a)$ is connected since $f(x) \leq a$ and $f(y) \leq a$ implies $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq a$, and $B(a)$ is closed since $f$ is continuous. Thus, we can write $B(a) = [l(a), r(a)]$ for any $a \in R$. $l(a) := 1$ and $r(a) := 0$ when $B(a) = \emptyset$. The functions $l$ and $r$ are right continuous since $[l(a), r(a)] = B(a) = \bigcup_{b > a} B(b) = \bigcup_{b > a} [l(b), r(b)] = \lim_{b \rightarrow a+} l(b), \lim_{b \rightarrow a+} r(b)$.

Let $c = \inf \{y : r(y) - l(y) \geq 1/2\}$, $a = l(c)$ and $b = r(c)$. Since $l$ and $r$ are right continuous, we have $b - a \geq 1/2$. In fact, $b - a = 1/2$ unless $f(x) = c$ for all $x \in [a, b]$. Suppose $b - a > 1/2$. Then $\{x : f(x) < c\} = \bigcup_{y < c} B(y) \subseteq [\inf_{y < c} l(y), \sup_{y < c} r(y)]$. $\sup_{y < c} r(y) - \inf_{y < c} l(y) \leq 1/2$ since for any $y_1, y_2 < c$, $r(y_1) - l(y_2) \leq r(\max(y_1, y_2)) - l(\max(y_1, y_2)) < 1/2$. Therefore, $\{x : f(x) = c\} = B(c) \setminus \{x : f(x) < c\}$ contains an interval of nonzero length. This combined with convexity implies $f \geq c$ hence $\underline{f} = 0$ on $B(c)$.

(1) When $f(x) = c$ for all $x \in [a, b]$

By convexity, we should have

$$\begin{align*}
&\begin{cases}
  c &\leq f(x) \leq 1 + \frac{c-1}{a}x &\text{if } 0 \leq x \leq a\\
  f(x) = c &\text{if } a \leq x \leq b\\
  c &\leq f(x) \leq 1 + \frac{c-1}{b}x &\text{if } b \leq x \leq 1
\end{cases}
\end{align*}$$

Therefore, $\int_a^b \frac{f(x) - c}{x} \leq \frac{1}{2} \int_a^b (1-c+\frac{c-1}{a})x + \frac{1}{2} \int_b^1 (1-c+\frac{c-1}{b})x - 1 - 1 = 1/4 < m$.

(2) When $b - a = 1/2$ and $a > 0$ and $b < 1$

We have $f(a) = f(b) = c$ since otherwise, $B(c) = \{x : f(x) \leq c\} = [a, b]$ should have been larger because of the intermediate value theorem. Then by convexity, we should have

- If $x < a$, then
  (a) $c = f(a) = f(\frac{b-a}{b-x}x + \frac{a-x}{b-x}b) \leq \frac{b-a}{b-x}f(x) + \frac{a-x}{b-x}c$, so $c \leq f(x)$
  (b) $f(x) = f(\frac{1-x}{a}x + \frac{x}{a}a) \leq (1 - \frac{x}{a})f(0) + \frac{x}{a}f(a) \leq 1 + \frac{1-c}{a}x$

- If $a < x < b$, then
  (a) $f(x) \leq c$
(b) \( c = f(a) = f\left(\left(1 - \frac{a}{2}\right) \ast 0 + \frac{a}{2} \ast x\right) \leq \left(1 - \frac{a}{2}\right) + \frac{a}{2} f(x) \), so \( 1 + \frac{c-1}{a} x \leq f(x) \)

(c) \( c = f(b) = f\left(\frac{1-b}{1-x} \ast x + \frac{b-x}{1-x} \ast 1\right) \leq \frac{1-b}{1-x} f(x) + \frac{b-x}{1-x}, \) so \( 1 + \frac{1-c}{1-b} (x-1) \leq f(x) \)

- if \( x > b \), then
  (a) \( c = f(b) = f\left(\frac{x-b}{x-a} a + \frac{b-a}{x-a} x\right) \leq \frac{x-b}{x-a} c + \frac{b-a}{x-a} f(x) \), so \( c \leq f(x) \)

(b) \( f(x) = f\left(\frac{1}{1-x} b + \frac{1}{1-x} \ast 1\right) \leq \frac{1}{1-x} c + \frac{1}{1-x} = 1 + \frac{1-c}{1-b} (x-1) \)

Therefore, \( \int_0^1 |f(x) - c| dx \leq \int_0^a (1-c+\frac{c-1}{a} x) dx + \int_a^b (c-\max(0,1+\frac{c-1}{a} x, 1+\frac{1-c}{1-b} (x-1))) dx + \int_b^1 (1-c+\frac{1-c}{1-b} (x-1)) dx \). A simple calculation shows that the RHS attains maximum when \( c = 1 - \sqrt{2} \) and the maximum is \( m = 1 - \frac{\sqrt{3}}{2} \).

(3) When \( b - a = 1/2 \) and either \( a = 0 \) or \( b = 1 \)

Almost the same consideration as the preceding case results in the desired upper bound

\[
f(x) = \begin{cases} 
1 - \sqrt{2} x & \text{if } x \leq \frac{\sqrt{2}}{2} \\
0 & \text{if } x > \frac{\sqrt{2}}{2}
\end{cases}
\]