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I claim that $m = 1 - \frac{\sqrt{2}}{2}$ is the minimum.

First, let's prove that $m = 1 - \frac{\sqrt{2}}{2}$ works. Let $f : [0,1] \to [0,1]$ be a convex function. f is continuous on $(0,1)$. We may assume that it is also continuous on 0 and 1, since we can let $f(0) := \sup_{0<\epsilon<1/2} \frac{f(1/2)-f(\epsilon)}{1/2-\epsilon}(0-1/2)+f(1/2)$ and $f(1) := \sup_{0<\epsilon<1/2} \frac{f(1-\epsilon)-f(1/2)}{1-\epsilon-1/2}(1-1/2)+f(1/2)$, while the integral doesn't change. Note that for any $a \in [0,1]$, $B(a) := \{x \in [0,1] : f(x) \le a\}$ is a closed interval if it is nonempty. This is true because $B(a)$ is connected since $f(x) \le a$ and $f(y) \le a$ implies $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \le a$, and $B(a)$ is closed since f is continuous. Thus, we can write $B(a) = [l(a), r(a)]$ for any $a \in R.(l(a) := 1$ and $r(a) := 0$ when $B(a) = \emptyset)$ The functions l and r are right continuous since $[l(a), r(a)] = B(a) = \bigcap_{b>a} B(b) = \bigcap_{b>a}[l(b), r(b)] = [\sup_{b>a} l(b), \inf_{b>a} r(b)] = [\lim_{b\to a+} l(b), \lim_{b\to a+} r(b)]$.

Let $c = \inf\{y : r(y) - l(y) \ge 1/2\}$, $a = l(c)$ and $b = r(c)$. Since l and r are right continuous, we have $b - a \ge 1/2$. In fact, $b - a = 1/2$ unless $f(x) = c$ for all $x \in [a,b]$. Suppose $b - a > 1/2$. Then $\{x : f(x) < c\} = \bigcup_{y<c} B(y) \subseteq [\inf_{y<c} l(y), \sup_{y<c} r(y)]$. $\sup_{y<c} r(y) - \inf_{y<c} l(y) \le 1/2$ since for any $y_1, y_2 < c$, $r(y_1) - l(y_2) \le r(\max(y_1,y_2)) - l(\max(y_1,y_2)) < 1/2$. Therefore, $\{x : f(x) = c\} = B(c) \setminus \{x : f(x) < c\}$ contains an intervel of nonzero length. This combined with convexity implies $f \ge c$ hence $\cancel{f = 0}$ on $B(c)$.

(1) When $f(x) = c$ for all $x \in [a,b]$
By convexity, we should have

$$\begin{cases} c \le f(x) \le 1 + \frac{c-1}{a}x & \text{if } 0 \le x \le a \\ f(x) = c & \text{if } a \le x \le b \\ c \le f(x) \le 1 + \frac{1-c}{1-b}(x-1) & \text{if } b \le x \le 1 \end{cases}$$

Therefore, $\int_0^1 |f(x)-c|dx \le \int_0^a (1-c+\frac{c-1}{a}x)dx + \int_b^1 1 - c + \frac{1-c}{1-b}(x-1)dx = \frac{a(1-c)}{2} + \frac{(1-b)(1-c)}{2} = \frac{(1-(b-a))(1-c)}{2} \le 1/4 < m$

(2) When $b - a = 1/2$ and $a > 0$ and $b < 1$
We have $f(a) = f(b) = c$ since otherwise, $B(c) = \{x : f(x) \le c\} = [a,b]$ should have been larger because of the intermediate value theorem. Then by convexity, we should have

- If $x < a$, then
  (a) $c = f(a) = f(\frac{b-a}{b-x}x + \frac{a-x}{b-x}b) \le \frac{b-a}{b-x}f(x) + \frac{a-x}{b-x}c$, so $c \le f(x)$
  (b) $f(x) = f((1-\frac{x}{a}) * 0 + \frac{x}{a} * a) \le (1-\frac{x}{a})f(0) + \frac{x}{a}f(a) \le 1 + \frac{c-1}{a}x$
- If $a < x < b$, then
  (a) $f(x) \le c$

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(b) $c = f(a) = f((1-\frac{a}{x})*0+\frac{a}{x}*x) \le (1-\frac{a}{x})+\frac{a}{x}f(x)$, so $1+\frac{c-1}{a}x \le f(x)$

(c) $c = f(b) = f(\frac{1-b}{1-x}*x+\frac{b-x}{1-x}*1) \le \frac{1-b}{1-x}f(x)+\frac{b-x}{1-x}$, so $1+\frac{1-c}{1-b}(x-1) \le f(x)$

- if $x > b$, then

(a) $c = f(b) = f(\frac{x-b}{x-a}a+\frac{b-a}{x-a}x) \le \frac{x-b}{x-a}c+\frac{b-a}{x-a}f(x)$, so $c \le f(x)$

(b) $f(x) = f(\frac{1-x}{1-b}*b+\frac{x-b}{1-b}*1) \le \frac{1-x}{1-b}c+\frac{x-b}{1-b} = 1+\frac{1-c}{1-b}(x-1)$

Therefore, $\int_0^1 |f(x)-c|dx \le \int_0^a(1-c+\frac{c-1}{a}x)dx+\int_a^b(c-\max(0,1+\frac{c-1}{a}x,1+\frac{1-c}{1-b}(x-1)))dx+\int_b^1(1-c+\frac{1-c}{1-b}(x-1))dx$. A simple calculation shows that the RHS attains maximum when $c = 1-\frac{\sqrt{2}}{2}$ and the maximum is $m = 1-\frac{\sqrt{2}}{2}$.

(3) When $b-a = 1/2$ and either $a = 0$ or $b = 1$

Almost the same consideration as the preceding case results in the desired upper bound

~~The extremal case is obtained when~~

$$\cancel{f(x) = \begin{cases} 1-\sqrt{2}x & \text{if } x \le \frac{\sqrt{2}}{2} \\ 0 & \text{if } x > \frac{\sqrt{2}}{2} \end{cases}}$$