2018 FALL PROBLEM OF THE WEEK
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We prove a more generalized result.

Problem. *Let p, q be two positive integers so that $p < q$. Then for every large n there is a graph G on n vertices so that G can be decomposed into q edge-disjoint spanning acylic subgraphs but cannot be decomposed into p edge-disjoint planar subgraphs.*

SOLUTION. Let $G(n, p_n)$ be the random graph on vertex set $\{1, \dots, n\}$ in which every pair $\{x, y\}$ with $x \neq y$ is an edge with probability p_n, independently and uniformly. We make use of the following result, which is fundamental in random graph theory[1].

Lemma 1. *For every $c > 1$, as $n \to \infty$ with probability $1 - o(1)$ $G(n, c/n)$ contains a connected component of size $yn - o(n)$ where $y = y(c)$ is the unique positive solution to the equation*

$$\exp(-cy) = 1 - y.$$

Note that $y \to 1^-$ as $c \to \infty$. We also need the following result.

Lemma 2. *Let $M \geq 3$ be a positive integer and H be a planar graph on n vertices containing no cycles of length smaller than M. Then H has at most $\frac{Mn}{M-2}$ edges.*

Proof. The lemma is true if $n = 1$. Assume that it is true for up to $n - 1$, we show that it is also true for n. Enough to prove when H is connected. If there is a vertex of degree at most 1, then simply delete it from H and apply induction hypothesis to the resulting graph to get

$$|E(H)| \leq \frac{M(n-1)}{M-2} + 1 \leq \frac{Mn}{M-2}.$$

If H has no vertex of degree at most 1 then assume furthermore that H is a plane graph, then since H is connected every face of H is bounded by a cycle (we count on the infinite face). Let e, f be the number of edges and faces of H, respectively. Then since H has no cycles of length smaller than M, we have $Mf \leq 2e$. By Euler's formula,

$$2 = n - e + f \leq n - e + \frac{2e}{M} = n - \frac{(M-2)e}{M},$$

which follows $e \leq \frac{Mn}{M-2}$ as desired. The proof is completed. $\square$

[1] See Chapter 11 in *The Probabilistic Method* by Alon and Spencer, 4th edition.

Coming back to the problem, let $c > 1$ be very large so that $y = y(c)$ in Theorem 1 is larger than p/q, and let M be very large so that

$$\frac{pM}{M-2} < qy. \tag{1}$$

Let $G_1, \dots, G_q$ be q independent elements of $G(n, c/n)$. For $i = 1, \dots, q$ let E_i be the event that G_i has no connected components of size at least $yn - o(n)$. By Theorem 1 we have

$$\mathbb{P}(E_i) = o(1) \quad \forall i = 1, \dots, q. \tag{2}$$

Let X be the number of cycles of length smaller than M in $\bigcup_{i=1}^q G_i$. Because every pair $\{x, y\}$ appears in $\bigcup_{i=1}^q G_i$ with probability at most qc/n, independently and uniformly, we have

$$E[X] \leq \sum_{k=3}^{M-1} \frac{n(n-1)\cdots(n-k+1)}{2k} \left(\frac{qc}{n}\right)^k \leq \sum_{k=3}^{M-1} \frac{(qc)^k}{2k} < \infty$$

so by Markov's inequality we have

$$\mathbb{P}(X \geq \log n) = o(1). \tag{3}$$

For $i \neq j \in \{1, \dots, q\}$, let $Y_{ij} = Y_{ji}$ be the number of common edges of G_i and G_j. We have

$$E[Y_{ij}] = \binom{n}{2} \left(\frac{c}{n}\right)^2 \leq \frac{c^2}{2}$$

so again by Markov's inequality

$$\mathbb{P}(Y_{ij} \geq \log n) = o(1) \quad \forall i, j = 1, \dots, q, i \neq j, \tag{4}$$

Combining (2), (3), and (4), we see that with probability $1 - o(1)$, each of $G_1, \dots, G_q$ has a connected component of size at least $yn - o(n)$, the union of them has at most $\log n$ cycles of length smaller than M, and any two of them have at most $\log n$ edges in common. By deleting edges if necessary we deduce that there exist forests $T_1, \dots, T_q$ on vertex set $\{1, \dots, n\}$ so that each has at least $yn - o(n) - q^2 \log n = yn - o(n)$ edges, the union of them has no cycles of length smaller than M, and any two of them are edge-disjoint. Let $G := \bigcup_{i=1}^q T_q$. Then G has at least $qyn - o(n)$ edges and no cycles of length smaller than M. Now suppose that G can be decomposed into p edge-disjoint planar graphs $H_1, \dots, H_p$, then by Lemma 2 we have

$$qyn - o(n) \leq |E(G)| = \sum_{j=1}^{p} |E(H_j)| \leq \frac{pMn}{M-2},$$

which is a contradiction for all large n, according to (1). The solution is completed. ❑