Problem. Suppose that the edges of a graph $G$ can be colored by 3 colors so that there is no monochromatic cycle. Prove or disprove that $G$ has two planar subgraphs $G_1$, $G_2$ such that $E(G) = E(G_1) \cup E(G_2)$.

Solution.

Suppose we have $G$ which satisfies $girth(G) \geq 6$ and $G_1$ and $G_2$ is planar subgraph of $G$. Since, $G_1$ and $G_2$ are subgraph of $G$, $girth(G_1) \geq 6$ and $girth(G_2) \geq 6$ also holds.

For any planar graph (possibly not connected) $v - e + f \geq 2$. $girth(H) \leq 6$ implies either $girth(H) = \infty$ or $6f \leq 2e$ since if there exists face, every face have larger than or equal to 6 edges, and edges are counted twice.

$girth(H) = \infty$ gives $e \leq v - 1$ and $6 \leq girth(H) < \infty$ gives, $f \leq \frac{e}{3}$. $2 \leq v - e + f \leq v - e + \frac{e}{3} \leq \frac{3}{2}(v - 2)$. So, $girth(G_1), girth(G_2) \geq 6$, $E(G_1) + E(G_2) \leq 2 \times \frac{3}{2}(V(G) - 2) = 3V(G) - 6$ if $V(G) \leq 4$.

$G$ can be colored by 3 colors implies that we can partition the edges of $G$ as 3 disjoint spanning forest. If all of those are tree then $E(G) = 3(V(G) - 1) = 3V(G) - 3 \geq 3V(G) - 6$, therefore $G$ cannot be partitioned as two planar subgraphs.

Thus, It is sufficient to find graph $G$ with following condition as counter-example:

- $G$ is union of three disjoint spanning trees.
- $girth(G) \leq 6$
We will construct a graph $G$ with $V(G) = 2p^2$, $E(G) = p^3$ and $girth(G) \leq 6$ for some prime $p$. ($p \geq 7$ will give $E(G) \leq 3V(G) - 3$, so we might find three disjoint spanning trees from this).

$H_p$ is bipartite graph where $V(H) = A \cup B$, $A = \{(x, y)_a|0 \leq x, y < p\}$, $B = \{(x, y)_b|0 \leq x, y < p\}$. And we will add edges connecting two vertices $(i, j)_a$ and $(k, (ik + j) \mod p)$ for all $i, j, k$ with $0 \leq i, j, k < p$. This trivially does not give odd length cycle, and also cycle of length 4. Proof is as follows:

Assume 4 different points $(i_1, j_1)_a$, $(i_2, j_2)_a$, $(k_1, s_1)_b$, $(k_2, s_2)_b$ forms a cycle.

Then, $s_1 \equiv i_1k_1 + j_1 \equiv i_2k_1 + j_2(\text{mod } p)$ and $s_2 \equiv i_1k_2 + j_1 \equiv i_2k_2 + j_2(\text{mod } p)$ Then $(i_1 - i_2)k_1 \equiv (j_2 - j_1)(\text{mod } p)$ and $(i_1 - i_2)k_2 \equiv (j_2 - j_1)(\text{mod } p)$

$i_1 = i_2$ implies $j_1 = j_2$ and it is contradiction to 4 different points. Therefore, $i_1 \neq i_2$, $k_1 = k_2 \equiv (j_2 - j_1)(i_1 - i_2)^{-1}(\text{mod } p)$, which gives $s_1 - s_2 \equiv i_1(k_1 - k_2) \equiv 0(\text{mod } p)$. This also leads to contradiction.

Suppose we choose all points with $(i - k) \equiv s - 1, s, s + 1(\text{mod } p)$ and make a subgraph. Then, There exists path $(i, j)_a - (i+s, i^2+s+i+j)_b - (i+1, j-i-s)_a - (i+s+1, i^2+(s+1)i+j+1)_b - (i, j+1)_a$ which means for any $i, j$, $(i, j)_a$ and $(i, j+1)_a$ is connected. $(i, *)_a$ is connected with $(i+1, *)_a$ and $(i, *)_a$ is connected with $(i+s, *)_b$. $gcd(s, p) = 1$ will give make this subgraph to connect every vertices of $G$, and there are spanning tree of this subgraph. This is illustrated as Binary adjacency matrix below.

We will construct this graph at $p = 11$. If we partition this graph with $(i - k) \equiv 0, 1, 2$ and $(i - k) \equiv 3, 4, 5$ and $(i - k) \equiv 6, 7, 8$; in each subgraph, there exist three disjoint spanning trees $T_1$, $T_2$, $T_3$.

$T_1 + T_2 + T_3$ is subgraph of $H_{11}$ so, $girth(T_1 + T_2 + T_3) \geq 6$.

This construction with graph $G = T_1 + T_2 + T_3$ with 242 vertices and 723 edges will let $G$ colored by 3 colors so that there is no monochromatic cycle with property that $G$ cannot be divided into two planar subgraph such that $E(G) = E(G_1) \cup E(G_2)$.

\[\square\]
Binary adjacency matrix of graph $H_5$.  
Edge is represented as 0.  
Four edges were chosen to show connectivity of $(3, 2)_a$ and $(3, 3)_a$