POW 2018-14 Forests and Planes

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Problem. Suppose that the edges of a graph G can be colored by 3 colors so that there is no monochromatic cycle. Prove or disprove that G has two planar subgraphs G_1, G_2 such that $E(G) = E(G_1) \cup E(G_2)$.

Solution.

Suppose we have G which satisfies $girth(G) \geq 6$ and G_1 and G_2 is planar subgraph of G. Since, G_1 and G_2 are subgraph of G, $girth(G_1) \geq 6$ and $girth(G_2) \geq 6$ also holds.

For any planar graph (possibly not connected) $v - e + f \geq 2$. $girth(H) \leq 6$ implies either $girth(H) = \infty$ or $6f \leq 2e$ since if there exists face, every face have larger than or equal to 6 edges, and edges are counted twice.

$girth(H) = \infty$ gives $e \leq v - 1$ and $6 \leq girth(H) < \infty$ gives, $f \leq \frac{e}{3}$, $2 \leq v - e + f \leq v - e + \frac{e}{3}$, $e \leq \frac{3}{2}(v - 2)$. So, $girth(G_1), girth(G_2) \geq 6$, $E(G_1) + E(G_2) \leq 2 \times \frac{3}{2}(V(G) - 2) = 3V(G) - 6$ if $V(G) \leq 4$.

G can be colored by 3 colors implies that we can partition the edges of G as 3 disjoint spanning forest. If all of those are tree then $E(G) = 3(V(G) - 1) = 3V(G) - 3 \geq 3V(G) - 6$, therefore G cannot be partitioned as two planar subgraphs.

Thus, It is sufficient to find graph G with following condition as counter-example:

- G is union of three disjoint spanning trees.
- $girth(G) \leq 6$

We will construct a graph G with $V(G) = 2p^2$, $E(G) = p^3$ and $girth(G) \leq 6$ for some prime p. ($p \geq 7$ will give $E(G) \leq 3V(G) - 3$, so we might find three disjoint spanning trees from this)

H_p is bipartite graph where $V(H) = A \cup B$, $A = \{(x, y)_a | 0 \leq x, y < p\}$, $B = \{(x, y)_b | 0 \leq x, y < p\}$. And we will add edges connecting two vertices $(i, j)_a$ and $(k, (ik + j) \mod p)$ for all i, j, k with $0 \leq i, j, k < p$. This trivially does not give odd length cycle, and also cycle of length 4. Proof is as follows:

Assume 4 different points $(i_1, j_1)_a$, $(i_2, j_2)_a$, $(k_1, s_1)_b$, $(k_2, s_2)_b$ forms a cycle.

Then, $s_1 \equiv i_1k_1 + j_1 \equiv i_2k_1 + j_2(\text{mod}p)$ and $s_2 \equiv i_1k_2 + j_1 \equiv i_2k_2 + j_2(\text{mod}p)$ Them $(i_1 - i_2)k_1 \equiv (j_2 - j_1)(\text{mod}p)$ and $(i_1 - i_2)k_2 \equiv (j_2 - j_1)(\text{mod}p)$

$i_1 = i_2$ implies $j_1 = j_2$ and it is contradiction to 4 different points. Therefore, $i_1 \neq i_2$, $k_1 = k_2 \equiv (j_2 - j_1)(i_1 - i_2)^{-1}(\text{mod}p)$, which gives $s_1 - s_2 \equiv i_1(k_1 - k_2) \equiv 0(\text{mod}p)$. This also leads to contradiction. $\square$

Suppose we choose all points with $(i - k) \equiv s - 1, s, s + 1(\text{mod}p)$ and make a subgraph. Then, There exists path $(i, j)_a - (i+s, i^2+si+j)_b - (i+1, j-i-s)_a - (i+s+1, i^2+(s+1)i+j+1)_b - (i, j+1)_a$ which means for any i, j, $(i, j)_a$ and $(i, j+1)_a$ is connected. $(i, *)_a$ is connected with $(i+1, *)_a$ and $(i, *)_a$ is connected with $(i+s, *)_b$. $gcd(s, p) = 1$ will give make this subgraph to connect every vertices of G, and there are spanning tree of this subgraph. This is illustrated as Binary adjacency matrix below.

We will construct this graph at $p = 11$. If we partition this graph with $(i - k) \equiv 0, 1, 2$ and $(i - k) \equiv 3, 4, 5$ and $(i - k) \equiv 6, 7, 8$; in each subgraph, there exist three disjoint spanning trees T_1, T_2, T_3.

$T_1 + T_2 + T_3$ is subgraph of H_{11} so, $girth(T_1 + T_2 + T_3) \geq 6$.

This construction with graph $G = T_1 + T_2 + T_3$ with 242 vertices and 723 edges will let G colored by 3 colors so that there is no monochromatic cycle with property that G cannot be divided into two planar subgraph such that $E(G) = E(G_1) \cup E(G_2)$. $\square$

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Binary adjacency matrix of graph H_5.
Edge is represented as `0`.
Four edges were chosen to show connectivity of $(3,2)_a$ and $(3,3)_a$