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2016____ Chae Jiseok

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We will call that the solution is trivial for the solution $x_1 = x_2 = \cdots = x_n = 0$.

The Diophantine equation of the case when $n = 1$ has only a trivial solution $x_1 = 0$ so we reject this case.

If $(x_1, x_2, \cdots, x_n)$ is a solution to the Diophantine equation then for any nonzero integer q, letting $p = \prod_{i=1}^{n} b_i$ we can see, by substituting, that $(x_1 q^{p/b_1}, x_2 q^{p/b_2}, \cdots, x_n q^{p/b_n})$ is also a solution. Therefore showing that the Diophantine equation has a nontrivial solution is sufficient to show that there are infinitely many solutions.

For the case $n = 2$ for convenience of notation we will solve the Diophantine equation $ax^k + by^m = 0$ where k, m are positive integers with $(k, m) = 1$, and a, b are nonzero integers. Not both k and m can be even, so we may assume m odd. By multiplying $\mathrm{sgn}(a)$ on both sides we may also assume that a is positive. Furthermore, since m is odd, by substituting $-y$ instead of y if necessary, we may solve the equation $ax^k = by^m$ assuming that b is also positive. We claim that there exists a nontrivial solution of the form $(x, y) = (a^\alpha b^\beta, a^\gamma b^\delta)$ where $\alpha, \beta, \gamma, \delta$ are all positive integers. Substituting such form gives us $a^{k\alpha+1} b^{k\beta} = a^{m\gamma} b^{k\delta+1}$. Therefore if $k\alpha + 1 = m\gamma$ and $k\beta = m\delta + 1$ then we are done. By Bézout's Identity we know that there exists an integer solution $\alpha = \alpha', \beta = \beta', \gamma = \gamma', \delta = \delta'$ for $k\alpha + 1 = m\gamma$ and $k\beta = m\delta + 1$. Then for any integers s and t we can see that $\alpha = \alpha' + ms, \beta = \beta' + mt, \gamma = \gamma' + ks, \delta = \delta' + kt$ is also a solution. Therefore taking s and t sufficiently large we may assume $\alpha', \beta', \gamma', \delta'$ all positive. Then as we observed, $(x, y) = \left(a^{\alpha'} b^{\beta'}, a^{\gamma'} b^{\delta'}\right)$ becomes a nontrivial integer solution of the Diophantine equation $ax^k = by^m$.

Now consider the case $n > 2$. By letting $x_2 = x_3 = \cdots = x_{n-1} = 0$ the Diophantine equation reduces to the case when $n = 2$. Since there exists a nontrivial integer solution for the reduced equation, we conclude that there also exists a nontrivial integer solution for the Diophantine equation $a_1 x_1^{b_1} + a_2 x_2^{b_2} + \cdots a_n x_n^{b_n} = 0$.

Therefore the given Diophantine equation $a_1 x_1^{b_1} + a_2 x_2^{b_2} + \cdots a_n x_n^{b_n} = 0$ has a nontrivial solution whenever $n > 1$. As discussed at the beginning, we conclude that the given Diophantine has infinitely many integer solutions.