POW 2018-11

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Fallacy

Problem. On a math exam, there was a question that asked for the largest angle of the triangle with sidelengths 21, 41, and 50. A student obtained the correct answer as follows: Let x be the largest angle. Then,

$$\sin x = \frac{50}{41} = 1 + \frac{9}{41}.$$

Since $\sin 90^\circ = 1$ and $\sin 12^\circ 40' 49'' = 9/41$, the angle $x = 90^\circ + 12^\circ 40' 49'' = 102^\circ 40' 49''$. Find the triangle with the smallest area with integer sidelengths and possessing this property (that the wrong argument as above gives the correct answer).

Solution. Consider a trinalge with integer sidelengths $a \le b \le c$ and the largest angle θ. The above condition is equivalent to

$$\theta = 90^\circ + \sin^{-1}\left(\frac{c}{b} - 1\right).$$

This makes sense since $1 \le \frac{c}{b} < \frac{a+b}{b} \le 2$. Moreover, it is equivalent to

$$\frac{c}{b} - 1 = \sin(\theta - 90^\circ) = -\cos\theta = \frac{c^2 - b^2 - a^2}{2ab}$$

which is rephrased as

$$a^2 + (b-a)^2 = (c-a)^2,$$

that is, $(x, y, z) := (a, b-a, c-a)$ forms a pythagorean triple. Note that the order of them is either $x < y < z$ or $y < x < z$. Now, let $A(x, y, z)$ be the area of this triangle. Then,

$$\begin{aligned}(2A(x,y,z))^2 &= a^2 b^2 \sin^2\theta = a^2 b^2 \left(1 - \cos^2\theta\right) = a^2 c(2b - c) \\ &= x^2(x+z)\left(y + (x + y - z)\right) > \left(\min\{x, y\}\right)^4.\end{aligned}$$

First check that $(2A(3,4,5))^2 = 432 < 5^4 < 720 = (2A(4,3,5))^2$. So, if $\min\{x,y\} \ge 5$, $A(x,y,z) > A(3,4,5)$. Since $\min\{x,y\} \ge 5$ for triples (x,y,z) other than (3,4,5) or (4,3,5), among all pythagorean triples, $A(3,4,5)$ is the smallest.

Therefore, the desired triangle is of sidelengths 3, 7, 8.