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Let **E** be the event where three pieces form an acute triangle.

Consider the stick as the closed interval $[0, 1]$ on the real line and x, y be two randomly chosen points on the interval.

We will first assume that $x < y$. Let a, b, and c be the lengths of three segments, then we without loss of generality have $a = x$, $b = y - x$, and $c = 1 - y$. Suppose that a, b, and c become the side lengths of an acute triangle. Let A be the angle opposite to the side with length a. Then by the law of cosines we have $\cos A = \frac{b^2+c^2-a^2}{2bc}$ where we want $\cos A > 0$ so that A is acute. That is, for the triangle to be acute, we should have $b^2 + c^2 > a^2$. For the same reason, we need the other two angles to be also acute, hence we also require $c^2 + a^2 > b^2$ and $a^2 + b^2 > c^2$.

Note that since b and c are both positive, we have $bc > 0$, hence if we have $b^2 + c^2 > a^2$ then we also automatically have $b + c = \sqrt{(b + c)^2} > \sqrt{b^2 + c^2} > \sqrt{a^2} = a$. Since a is also positive, we have $ca > 0$ and $ab > 0$ also. Thus by the same logic, if $c^2 + a^2 > b^2$ and $a^2 + b^2 > c^2$ we also have $c + a > b$ and $a + b > c$. Since $a + b > c$, $b + c > a$, and $c + a > b$ are the conditions for three line segments of length a, b, and c to form a triangle, we get that once a, b, and c satisfy the three inequalities $b^2 + c^2 > a^2$, $c^2 + a^2 > b^2$, and $a^2 + b^2 > c^2$ then it is provided that a, b, and c become the side lengths of a triangle.

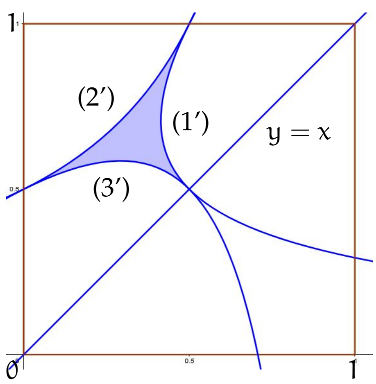
Expressing the inequalities in terms of x and y and arranging the terms, we have

$$b^2 + c^2 > a^2 \iff 1 - 2y - 2xy + 2y^2 > 0 \iff \frac{1 - 2y + 2y^2}{2y} > x \tag{1}$$

$$c^2 + a^2 > b^2 \iff 1 - 2y > -2xy \iff \frac{1}{2 - 2x} > y \tag{2}$$

$$a^2 + b^2 > c^2 \iff 2x^2 - 2xy > 1 - 2y \iff y > \frac{1 - 2x^2}{2 - 2x} \tag{3}$$

and plotting the regions defined by (1), (2), and (3) with $y > x$ we have the shaded region in the following figure. In the figure, (1′) indicates the boundary curve of the region corresponding to (1), (2′) that of (2), and (3′) that of (3).



Since no two angles of a triangle can be both obtuse, the boundary curves do not intersect inside the upper triangle. By substituting $(x, y) = (1 - \tilde{y}, 1 - \tilde{x})$ we can easily check that the figure is symmetric along the line $x+y=1$. Also by substituting, we can see that the boundary curves of (1) and (2) intersect at $(\frac{1}{2},1)$, (2) and (3) at $(0,\frac{1}{2})$, and (3) and (1) at $(\frac{1}{2},\frac{1}{2})$.

The common region of (1), (2), (3), and $y>x$ is the shaded region in the figure. Using the symmetry of (1) and (3) along the line $x+y=1$, we can compute the area of the shaded region as

$$\begin{aligned}
& \frac{1}{2}-\int_0^{1/2}\left(1-\frac{1}{2-2x}\right)dx-2\int_0^{1/2}\left(\frac{1-2x^2}{2-2x}-x\right)dx \\
= & \frac{1}{2}-\frac{1}{2}\int_0^{1/2}\left(2-\frac{1}{1-x}\right)dx-\int_0^{1/2}\left(2-\frac{1}{1-x}\right)dx \\
= & -1+\frac{3}{2}\int_0^{1/2}\frac{1}{1-x}dx \\
= & -1+\frac{3}{2}\Big[-\ln(1-x)\Big]_0^{1/2} \\
= & -1+\frac{3}{2}\ln 2.
\end{aligned}$$

The region corresponding to the event where $x<y$ is the triangular region bounded by $x=0$, $y=1$, and $x=y$, which has area $1/2$. Therefore the probability of **E** occurring given that $x<y$ is

$$P(\mathbf{E}\mid x<y)=\frac{-1+(3/2)\ln 2}{1/2}=-2+3\ln 2.$$

All of the logic up to this point can be applied to the case where $x>y$ with the role of x and y exchanged. Therefore we also have that

$$P(\mathbf{E}\mid x>y)=-2+3\ln 2.$$

Since x and y are randomly chosen from the interval $[0,1]$, we know that $P(x=y)=0$ and $P(x<y)=P(x>y)=1/2$. Hence we obtain

$$\begin{aligned}
P(\mathbf{E}) &= P(x<y)\,P(\mathbf{E}\mid x<y)+P(x=y)\,P(\mathbf{E}\mid x=y)+P(x>y)\,P(\mathbf{E}\mid x>y) \\
&= \frac{1}{2}(-2+3\ln 2)+0+\frac{1}{2}(-2+3\ln 2) \\
&= -2+3\ln 2.
\end{aligned}$$

Therefore the probability that three pieces form an acute triangle is $-2+3\ln 2$.