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2014**** Lee, Jongwon

Let $a_n, a_{n-1}, \ldots, a_0$ be any positive integers and let $f(x) = a_n x^n + \cdots + a_0$. Firstly, we shall see that $S(f(10)) \leq f(1)$. Note that if $a_i < 10$ for all i, then we have $S(f(10)) = f(1)$.

The following is the usual algorithm for adding numbers:
Starting from $i = 0$, if $a_i \geq 10$, then decrease a_i by 10 and increase a_{i+1} by 1. Repeat this until a_i becomes less than 10 and then move onto next i.

Since every iteration decreases $f(1)$ by 9, keeps $f(10)$ the same, and eventually all coefficients become < 10, we have the desired result. Furthermore, from the algorithm, we have that $S(f(10)) = f(1)$ if and only if $a_i < 10$ for all i.

Back to the original problem, if $n = a_n a_{n-1} \cdots a_0$ and $f(x) = a_n x^n + \cdots + a_0$, we have $S(n^r) = S(f(10)^r)$ and $S(n)^r = f(1)^r$. Applying the above paragraph with $f(x)^r$, we have that $S(n^r) = S(n)^r$ if and only if all coefficients of $f(x)^r$ are less than 10.

I don't have a *complete* description of such (n, r)'s, but the condition excludes many possibilities as follows:

First, note that if $n = 10^k$ for some k, then result holds for any r. Also the result hols for any n if $r = 1$.

Now assume that $r > 1$ and that there is some i with $a_i \geq 2$ or there are some $i \neq j$ with $a_i, a_j \geq 1$. If $r \geq 5$, then in the first case, we have $a_i^r \geq 32$, so it is impossible, and in the second case, we have $\binom{r}{2} a_i^2 a_j^{r-2} \geq 10$, so it is impossible. Therefore, we must have $r \leq 4$.

Looking at the case $r = 4$, if $a_i \geq 2$ for some i, then $a_i^r \geq 10$ so it is impossible. And if $a_i, a_j, a_k \geq 1$ for some distinct i, j, k, then $\binom{r}{r-2,1,1} a_i^{r-2} a_j a_k \geq 12$, so it is also impossible. Therefore, if $r = 4$, we must have $n = 10^k + 10^l$ for some distinct k, l, or $n = 10^k$ for some k.

If $r = 3$, since $a_i^3 < 10$ for all i, we have $a_i \leq 2$ for all i. If $a_i = 2$ for some i and $a_j \geq 1$ for some $j \neq i$, then we have $3a_i^2 a_j \geq 12$, which is impossible. Therefore, if $a_i = 2$ for some i, then we must have $n = 2 \times 10^i$. For the case where $a_i \leq 1$ for all i, the assertion of the result depends on some properties of the set $B = \{i : a_i \neq 0\}$. For example, it is great if $B + B + B$, as a multiset, does not contain duplicate elements.

If $r = 2$, since $a_i^2 < 10$ for all i, we have $a_i \leq 3$ for all i. If $a_i = 3$ for some i and $a_j \geq 2$ for some $j \neq i$, then $2a_i a_j \geq 12$ so it is impossible. Therefore, if $a_i = 3$ for some i, then $a_j \leq 1$ for all $j \neq i$. Other restrictions again depend on the set $B = \{i : a_i \neq 0\}$ and on the multiset $B + B$.