POW 2018-07

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Consider a vector

$$v = \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{C}^n$$

as a polynomial

$$f(x) = c_0 + c_1 x^1 + \cdots + c_n x^n$$

Then,

$$S^T v = \begin{bmatrix} c_1 \\ nc_0 + 2c_2 \\ \cdots \\ 2c_{n-2} + nc_n \\ c_{n-1} \end{bmatrix}$$

corresponds to

$$(c_1 + 2c_2 x + \cdots) + (nc_0 x + (n-1)c_1 x^2 + \cdots) = f'(x) + (nxf(x) - x^2 f'(x))$$

If we let $g_k(x) = (1-x)^k(1+x)^{n-k}$, then we have

$g_k(x)$

$$\begin{aligned} & g_k'(x) + nxg_k(x) - x^2 g_k'(x) \\ &= (1-x^2)(-k(1-x)^{k-1}(1+x)^{n-k} + (n-k)(1-x)^k(1+x)^{n-k-1}) + nxf(x) \\ &= (1-x)^k(1+x)^{n-k}(-k(1+x) + (n-k)(1-x) + nx) = (n-2k)f(x) \end{aligned}$$

Therefore, varying k in $0, 1, \ldots, n$, we see that g_k correspond to an eigenvector of the eigenvalue $n - 2k$. (It is clear that $g_k \neq 0$.) In particular, we have found all $n+1$ eigenvalues of S^T, thus that of S, which are $-n, -n+2, \ldots, n$.