POW 2018-05

2016____ Chae Jiseok

March 32, 2018

For convenience we let $p = 18/38$ and $q = 1 - p = 20/38$. For integers $0 \leqslant k \leqslant 40$, let P_k be the probability such that the player, having \$k, reaches \$40 before he goes bankrupt. We see that $P_0 = 0$ since it is the case where the player is bankrupt hence cannot continue gambling, and $P_{40} = 1$ since it is the case where the player has already reached his goal. Note that P_k only depends on how much money the player has, regardless of how long the player has been playing roulette. Then for integers $1 \leqslant k < 40$ we have the following relation

$$P_k = pP_{k+1} + qP_{k-1}$$

since each time the player bets a dollar, the player will earn \$1 and become having $\$(k+1)$ with probability p, or lose \$1 and become having $\$(k-1)$ with probability q. With the observation that $p + q = 1$ we can rewrite the relation in the following form

$$(p+q)P_k = pP_{k+1} + qP_{k-1} \qquad \Rightarrow \qquad P_{k+1} - P_k = \frac{q}{p}(P_k - P_{k-1}).$$

Then we can observe that

$$\begin{aligned} P_2 - P_1 &= \frac{q}{p}(P_1 - P_0) &&= \frac{q}{p}P_1 \\ P_3 - P_2 &= \frac{q}{p}(P_2 - P_1) &&= \left(\frac{q}{p}\right)^2 P_1 \\ &\vdots \\ P_{k+1} - P_k &= \frac{q}{p}(P_k - P_{k-1}) &&= \left(\frac{q}{p}\right)^k P_1 \end{aligned}$$

which, with telescoping sum and geometric series, implies

$$P_k = P_1 + \sum_{i=1}^{k-1}(P_{i+1} - P_i) = P_1 + \sum_{i=1}^{k-1}\left(\frac{q}{p}\right)^i P_1 = \sum_{i=0}^{k-1}\left(\frac{q}{p}\right)^i P_1 = \frac{1-(q/p)^k}{1-q/p}P_1.$$

Plugging in $k = 40$ implies

$$1 = P_{40} = \frac{1-(q/p)^{40}}{1-q/p}P_1$$

hence

$$\begin{aligned} P_{20} = \frac{1-(q/p)^{20}}{1-q/p}P_1 = \frac{1-(q/p)^{20}}{1-q/p}\cdot\frac{1-q/p}{1-(q/p)^{40}} &= \frac{1-(q/p)^{20}}{1-(q/p)^{40}} \\ &= \frac{1-(20/18)^{20}}{1-(20/18)^{40}} \end{aligned}$$

which is by definition the probability of player starting at \$20 reaching \$40 without going bankrupt.