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2014**** Lee, Jongwon

The answer is $n + \cos(2\pi/n)$. Let $\omega = 2\pi/n$. Define $e_0, \ldots, e_{n-1} \in \mathbb{C}^n$ as

$$e_k = \frac{1}{\sqrt{n}}(1, e^{ik\omega}, e^{2ik\omega}, \ldots, e^{(n-1)ik\omega})$$

Then, these vectors form an orthonormal basis. Define $c_0, \ldots, c_{n-1} \in \mathbb{C}$ as

$$v := (x_1, \ldots, x_n) = c_0 e_0 + \cdots + c_{n-1} e_{n-1}$$

The first constrains becomes

$$n = x_1 + \ldots + x_n = \sqrt{n} c_0 \implies c_0 = \sqrt{n}$$

since the sum of the components of e_k vanishes for $k \neq 0$ and equals $\sqrt{n}$ for $k = 0$. The second becomes

$$\begin{aligned} n + 1 = x_1^2 + \ldots + x_n^2 = (v, v) = |c_0|^2 + \cdots + |c_{n-1}|^2 \\ \implies |c_1|^2 + \cdots + |c_{n-1}|^2 = 1 \end{aligned}$$

where $(-, -)$ denotes the usual inner product in $\mathbb{C}^n$. Also, that $x_1, \ldots, x_n \in \mathbb{R}$ is equivalent to

$$\begin{aligned} v = \overline{v} &\implies c_0 e_0 + \sum_{k=1}^{n-1} c_k e_k = c_0 e_0 + \sum_{k=1}^{n-1} \overline{c_k e_k} = c_0 e_0 + \sum_{k=1}^{n-1} \overline{c}_k e_{n-k} \\ &\implies c_k = \overline{c_{n-k}} \quad \forall 1 \leq k \leq n-1 \end{aligned}$$

Finally, if A is a linear transformation on $\mathbb{C}^n$ defined as $A(a_1, \ldots, a_n) = (a_2, \ldots, a_n, a_1)$, then we have $Ae_k = e^{ik\omega} e_k$, so that

$$\begin{aligned} &x_1 x_2 + \cdots + x_n x_1 \\ &= (v, Av) \\ &= n + \sum_{k=1}^{n-1} e^{ik\omega} |c_k|^2 \\ &= n + \sum_{k=1}^{n-1} e^{ik\omega} \frac{|c_k|^2 + |c_{n-k}|^2}{2} \\ &= n + \sum_{k=1}^{n-1} \frac{e^{ik\omega} + e^{-ik\omega}}{2} |c_k|^2 \\ &\leq n + \max_{1 \leq j \leq n-1} (\cos j\omega) \sum_{k=1}^{n-1} |c_k|^2 = n + cos\omega \end{aligned}$$

The equality holds if and only if $c_1 = c_{n-1} = \sqrt{1/2}$, in other words, $x_j = 1 + \sqrt{2/n}\cos(j\omega)$ for all j.