KAIST POW 2018-02

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Problem. *For $n \geq 1$, let $f(x) = x^n + \sum_{k=0}^{n-1} a_k x^k$ be a polynomial with real coefficients. Prove that if $f(x) > 0$ for all $x \in [-2, 2]$, then $f(x) \geq 4$ for some $x \in [-2, 2]$.*

Solution. We consider a sequence of polynomials $\{p_n\}$ defined as

$$p_0(x) = 1, p_1(x) = x, p_{n+2}(x) = 2xp_{n+1}(x) - p_n(x).$$

Because of the formula $\cos(n+2)\theta = 2\cos\theta\cos(n+1)\theta - \cos n\theta$, we note that $p_n(\cos\theta) = \cos(n\theta)$. Thus, taking $t_i = \cos(i\pi/n)$ for $i = 0, \cdots, n$, we have $1 = t_0 > t_1 > \cdots > t_n = -1$ and $p_n(t_i) = (-1)^i$. Note also that $p_n(x)$ is a polynomial of degree n with the coefficient of highest term 2^{n-1} for $n \geq 1$. Now we define $q_n(x) = 2p_n(x/2)$: it is a monic polynomial of degree n with $2 = 2t_0 > 2t_1 > \cdots > 2t_n = -2$ satisfying $q_n(2t_i) = 2(-1)^i$. Now, suppose f with $M = \max_{[-2,2]} f < 4$ satisfies the condition of the problem. Then, we define $g = f - M/2$. We have $-2 < -M/2 \leq g(x) \leq M/2 < 2$ on $[-2, 2]$, so $g(2t_i) - q_n(2t_i) < 0$ for even i and $g(2t_i) - q_n(2t_i) > 0$ for odd i. By intermediate value theorem, there exists $s_1, \cdots, s_n$ with $2t_{i-1} < s_i < 2t_i$ satisfying $g(s_i) - q_n(s_i) = 0$. However, $g - q_n$ is a polynomial of degree less than n, so having n distinct zeroes imply that $g - q_n = 0$. However, $-2 < g(x) < 2$ does not hold on $[-2, 2]$ in this case, leading to contradiction. ■