POW 2018-01

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March 7, 2018

Define

$$p_n = \prod_{k=1}^{n-1} \frac{2k+1}{k} \quad n > 1$$

and $p_1 = 1$. Then

$$\frac{a_n}{p_n} = \frac{a_{n-1}}{p_{n-1}} - \frac{1}{p_n} \Rightarrow a_n = p_n \left(a - \sum_{i=2}^{n} \frac{1}{p_i} \right)$$

Since $p_n > 2^{n-1}$, $\sum_{i=2}^{\infty} \frac{1}{p_i}$ converges.

Lemma 0.1. $\sum_{i=2}^{\infty} \frac{1}{p_i} = \frac{\pi}{2} - 1$

Proof.

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1} x dx = \frac{2(n-1)}{2n-1} \int_0^{\frac{\pi}{2}} \sin^{2n-3} x dx \Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2n-1} x dx = \frac{2^{n-1}}{p_n} \int_0^{\frac{\pi}{2}} \sin^1 x dx = \frac{2^{n-1}}{p_n}$$

Since $\left| \frac{\sin^{2n-1} x}{2^{n-1}} \right| < \frac{1}{2^{n-1}}$ and $\sum_{n \geq 2} \frac{1}{2^{n-1}}$ converge, $\sum_{n \geq 2} \frac{\sin^{2n-1} x}{2^{n-1}}$ converge uniformly. So

$$\sum_{n \geq 2} \frac{1}{p_n} = \int_0^{\frac{\pi}{2}} \sum_{n \geq 2} \frac{\sin^{2n-1} x}{2^{n-1}} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{2 - \sin^2 x} dx = \int_0^1 \frac{1-x^2}{1+x^2} dx = \frac{\pi}{2} - 1$$

□

If $a - \sum_{i=2}^{\infty} \frac{1}{p_i} \neq 0$ then

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} p_n \left(a - \sum_{i=2}^{n} \frac{1}{p_i} \right) = \infty$$

since $\lim_{n \to \infty} p_n = \infty$.
So let's assume $a = \sum_{i=2}^{\infty} \frac{1}{p_i}$ then

$$a_n = p_n \sum_{i \geq 1}^{\infty} \frac{1}{p_{n+i}} = \sum_{i \geq 1} \frac{p_n}{p_{n+i}} = \sum_{i \geq 0} \prod_{k=0}^{i} \frac{1}{\frac{2(n+k)+1}{n+k}}$$

From this we have following facts.So $\{a_n\}$ is a increasing bounded above sequence. So $\lim_{n \to \infty} a_n$ converges only when $a = \frac{\pi}{2} - 1$.