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Integers from square roots

Problem. Find all integers n such that $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}$ is an integer.

Lemma. For any $Q \in \mathbb{Z}[x]$ and $n \in \mathbb{N}$, $Q(x-\sqrt{n})\,Q(x+\sqrt{n}) \in \mathbb{Z}[x]$.

proof. If $Q(x)=\sum_i a_i x^i$, then $Q(s)Q(t)=\sum_i {a_i}^2 (st)^i + \sum_{i<j} a_i a_j (st)^i \left(s^{j-i}+t^{j-i}\right)$. Since both $(x-\sqrt{n})(x+\sqrt{n})$ and $(x-\sqrt{n})^{j-i}+(x+\sqrt{n})^{j-i}$ are in $\mathbb{Z}[x]$, so is $Q(x-\sqrt{n})\,Q(x+\sqrt{n})$. $\square$

Solution. $n=1$ is a trivial solution. Suppose there exists a solution $n>1$ and pick the smallest one. From the lemma, define $\{P_k\} \subseteq \mathbb{Z}[x]$ such that

$$\begin{aligned}&P_0(x)=x \text{ and for } k \geq 1,\\ &P_k(x)=P_{k-1}\left(x-\sqrt{k}\right)P_{k-1}\left(x+\sqrt{k}\right).\end{aligned}$$

This is represented in another way (product of alternating signs) as

$$P_k(x)=\prod_{j=0}^{2^k-1}\left\{x+\sum_{i=1}^{k}(-1)^{\left\lfloor \frac{j}{2^{i-1}}\right\rfloor}\sqrt{i}\right\}.$$

Let $S_k=\sum_{i=1}^{k}\sqrt{i}$ and note that $P_k(S_k)=0$ for all k. Now we consider minimal polynomial x^2-n of both $\sqrt{n}$ and $-\sqrt{n}$ over $\mathbb{Z}$. Since $P_{n-1}(S_n-\sqrt{n})=0$ and $S_n \in \mathbb{Z}$, $x^2-n \mid P_{n-1}(S_n-x) \in \mathbb{Z}[x]$.

$$\begin{aligned}\left|P_{n-1}\left(S_n+\sqrt{n}\right)\right| &\geq \prod_{j=0}^{2^{n-1}-1}\left\{\left|S_n+\sqrt{n}\right|-\left|\sum_{i=1}^{n-1}(-1)^{\left\lfloor \frac{j}{2^{i-1}}\right\rfloor}\sqrt{i}\right|\right\}\\ &\geq \prod_{j=0}^{2^{n-1}-1}\left\{\left|S_n+\sqrt{n}\right|-\left|S_{n-1}\right|\right\}>0.\end{aligned}$$

However, $P_{n-1}(S_n+\sqrt{n}) \neq 0$. So it's a contradiction. $\square$