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Lemma. Let $x_1, \dots, x_n, y$ be rational numbers such that there is no relation as

$$y = r^2 x_1^{a_1} \cdots x_n^{a_n} \tag{1}$$

for some integers $a_1, \dots, a_n$ and some rational r. Then, $\sqrt{y} \notin \mathbb{Q}(\sqrt{x_1}, \dots, \sqrt{x_n})$, i.e. $\sqrt{y}$ cannot be represented as a rational polynomial of $\sqrt{x_1}, \dots, \sqrt{x_n}$.

Proof. We shall use mathematical induction on n. If $n = 0$, the condition on y is that y is not a square of a rational number. So $\sqrt{y_i} \notin \mathbb{Q}$.

Suppose that the statement holds for all numbers $< n$. For $0 \le i \le n$, let $K_i = \mathbb{Q}(\sqrt{x_1}, \dots, \sqrt{x_i})$. If $K_{n-1} = K_n$, then by induction hypothesis, we have $\sqrt{y} \notin K_{n-1} = K_n$.

Now assume that $K_{n-1} \neq K_n$ and that $\sqrt{y} \in K_n$. Since K_n is a degree 2 extension of K_{n-1}, we have

$$A\sqrt{x_n} + B = \sqrt{y}$$

for some $A, B \in K_{n-1}$. Then,

$$(A^2 x_n + B^2 - y) + 2AB\sqrt{x_n} = 0$$

so we should have $A^2 x_n + B^2 - y = 0$ and $2AB = 0$, since otherwise we would have had $K_{n-1} = K_n$. If $A = 0$, then we have $B^2 = y$, which contradicts the induction hypothesis $\sqrt{y} \notin K_{n-1}$. If $B = 0$, then we have $A^2 x_n = y$, so that $\sqrt{x_n y} \in K_{n-1}$. By the (contrapositive of the) induction hypothesis, there must be a relation

$$x_n y = r^2 x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$$

which is a contradiction to (1). This finishes the proof of the lemma. □

Back to the original problem, let E be the set of integers from 1 to n whose exponent of 2 in its prime factorization is even. Then, we have

$$\sqrt{1} + \cdots + \sqrt{n} = \sqrt{2}\left(\sum_{f \notin E} \sqrt{\frac{f}{2}}\right) + \sum_{e \in E} \sqrt{e}$$

Since $f/2 \in E$ for $f \notin E$, if the above value is an integer, we have that

$$\sqrt{2} \in \mathbb{Q}(\sqrt{e_1}, \dots, \sqrt{e_m})$$

where $e_1, \dots, e_m$ is just an enumeration of elements of E. But then, it is a contradiction to the lemma since

$$2 = r^2 e_1^{a_1} \cdots e_m^{a_m}$$

is impossible because the exponent of 2 in the rhs will always be even.