KAIST POW 2017-18 : Limit

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Let $\epsilon > 0$. From the definition of the limit, there exists $N \in \mathbb{R}$ that

$$2 - \epsilon < f(x) + f'(x) < 2 + \epsilon$$

holds for all $x > N$.

Now for every $x > N$, $f(x) < 2 - 2\epsilon$ implies $f'(x) > \epsilon$: if there exists $x_0 > N$ that $f(x_0) < 2 - 2\epsilon$, Then there exists $x_1 > N$ that $f(x_1) = 2 - 2\epsilon$, since otherwise $f(x) < 2 - 2\epsilon$ hence $f'(x) > \epsilon$ for every $x > x_0$ then f diverges to infinity, which implies a contradiction. Then we may show following lemma.

Lemma 1. $f(x) \geq 2 - 2\epsilon$ for all $x \geq x_1$.

Proof. Suppose not. then there exists $x_2 > x_1$ that $f(x_2) < 2 - 2\epsilon$. Then f has lower bound $f(x_3)$ in the compact interval $[x_1, x_2]$: if $f(x_3) \leq f(x_2)$ and $x_3 \neq x_2$, then $f'(x_3) = 0$ and $f(x_3) < 2 - 2\epsilon$ hence their sum is clearly smaller than $2 - \epsilon$. Only possible case is the case that $f(x_2)$ is minimum at that interval, but in that case $f'(x_2)$ should be nonpositive hence the sum still lies below $2 - \epsilon$. □

Hence we have $\liminf_{x \to \infty} f(x) \geq 2 - 2\epsilon$ if such x_0 exists. But $\liminf_{x \to \infty} f(x) \geq 2 - 2\epsilon$ holds if such x_0 does not exist, since then $f(x) \geq 2 - 2\epsilon$ for all $x > N$.

By same argument we can show that $\limsup_{x \to \infty} f(x) \leq 2 + 2\epsilon$. Taking $\epsilon \to 0^+$ implies the answer : the limit of f exist and its value equals 2.