POW 2017-17 An Infimum 2016 최대범

By Lagrange Identity, for $a_k, b_k \in \mathbb{R}$,

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \left(\sum_{k=1}^{n} a_k b_k\right)^2 + \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2$$

$$\ge \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2 .$$

Let $a_k = c_k$, $b_k = c_k d_k$ for some $c_k, d_k \in \mathbb{R}$. Then

$$\sum_{k=1}^{n} c_k^2 \cdot \sum_{k=1}^{n} c_k^2 d_k^2 \ge \sum_{1\le k<j\le n} c_k^2 c_j^2 (d_k - d_j)^2$$

$$\ge \sum_{1\le k\le n-1} c_k^2 c_{k+1}^2 (d_k - d_{k+1})^2 + c_n^2 c_1^2 (d_1 - d_n)^2$$

Since $n \ge 3$.

alpha_k ≤ alpha_{k+1}

Let $c_k = \dfrac{1}{\alpha_k - \alpha_{k+1}}$, $d_k = \alpha_k$ for some ~~distinct~~ $\alpha_k \in \mathbb{R}$, $1 \le k \le n$,

and Let $\alpha_{n+1} = \alpha_1$, $c_{n+1} = c_1$, $d_{n+1} = d_1$. ~~Then~~, $d_{n+2} = d_2$. Then

$$\sum_{k=1}^{n}\left(\frac{1}{\alpha_k - \alpha_{k+1}}\right)^2 \cdot \sum_{k=1}^{n} \frac{\alpha_k^2}{(\alpha_k - \alpha_{k+1})^2} \ge \sum_{k=1}^{n}\left(\frac{1}{\alpha_k - \alpha_{k+1}}\right)^2\left(\frac{1}{\alpha_{k+1} - \alpha_{k+2}}\right)^2 \cdot (\alpha_k - \alpha_{k+1})^2$$

$$= \sum_{k=1}^{n}\left(\frac{1}{\alpha_{k+1} - \alpha_{k+2}}\right)^2 = \sum_{k=1}^{n}\left(\frac{1}{\alpha_k - \alpha_{k+1}}\right)^2 .$$

So $\displaystyle\sum_{k=1}^{n} \frac{\alpha_k^2}{(\alpha_k - \alpha_{k+1})^2} \ge 1$.

Since $\displaystyle\prod_{k=1}^{n} x_k = 1$, $x_k \ne 1$, $x_k = \dfrac{\alpha_k}{\alpha_{k+1}}$ for some distinct $\alpha_k \in \mathbb{R}$,

$\alpha_{n+1} = \alpha_1$. So

$$\sum_{k=1}^{n} \frac{x_k^2}{(1 - x_k)^2} = \sum_{k=1}^{n} \frac{\alpha_k^2}{(\alpha_k - \alpha_{k+1})^2} \ge 1 .$$

Then infimum is ≥ 1.

Let $x_1 = m^{n-1}$, $x_2 = \cdots = x_n = \dfrac{1}{m}$ for some $m \in \mathbb{N}$.

Then $\displaystyle\sum_{k=1}^{n} \frac{x_k^2}{(1-x_k)^2} = \frac{(m^{n-1})^2}{(1-m^{n-1})^2} + (n-1)\frac{(\frac{1}{m})^2}{(1-\frac{1}{m})^2} \longrightarrow 1$ as $m \to \infty$.

So infimum is exactly 1.