KAIST POW 2017-15 : Infinite product

Ki Joung Jang

September 24, 2017

(Existence) Given $x_1, \cdots, x_n (n \geq 0)$, let x_{n+1} be the smallest positive integer that

$$\prod_{i=1}^{n+1} \left(1 + \frac{1}{x_i}\right) < x.$$

It is trivial that such integer exists, moreover we have $x_{n+1} \geq x_n^2 (n \geq 1)$, since otherwise

$$\left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n+1}}\right) \geq \left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_n^2 - 1}\right) = 1 + \frac{1}{x_n - 1}$$

hence

$$\prod_{i=1}^{n-1} \left(1 + \frac{1}{x_i}\right)\left(1 + \frac{1}{x_n - 1}\right) < x$$

so $x_n - 1$ would be chosen instead of x_n.

To prove the convergence of product, define $p_n = \prod_{i=1}^{n} (1 + 1/x_i)$. Since we have the inequality

$$p_n = p_{n-1}\left(1 + \frac{1}{x_n}\right) < x \leq p_{n-1}\left(1 + \frac{1}{x_n - 1}\right)$$

, hence we have

$$0 < x - p_n \leq \frac{p_{n-1}}{x_n(x_n - 1)} < \frac{2}{x_n(x_n - 1)}.$$

Since $x_n \to \infty$ from the fact $x_1 \geq 2$ and $x_{i+1} \geq x_i^2$, the existence is proved.

(Uniqueness) Suppose that there are two different sequences $\{x_i\}$, $\{y_i\}$ which satisfies same property for given $x \in (1, 2)$. Then we can set minimal index l that x_l and y_l differs for the first time. WLOG assume $x_l < y_l$. Let $p = \prod_{i=1}^{l-1}(1 + 1/x_i)$. Then

$$\begin{aligned} x &> p\left(1 + \frac{1}{x_l}\right) \\ &\geq p\left(1 + \frac{1}{y_l - 1}\right) \\ &= p\prod_{i=0}^{\infty}\left(1 + \frac{1}{y_l^{2^i}}\right) \\ &\geq \prod_{i=1}^{\infty}\left(1 + \frac{1}{y_i}\right) = x \end{aligned}$$

, which is a contradiction.