2017 FALL PROBLEM OF THE WEEK
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For all nonnegative integer n, call $\mathbb{R}^n[x]$ the vector space consisting of real polynomials in variable x of degree not exceeding n; call $T_n(x)$ and $U_n(x)$, respectively, the Chebyshev polynomials of degree n of first and second kind. For all $i = \overline{1,n}$ denote $x_i := \cos(\frac{2i-1}{2n}\pi)$, so $x_1, x_2, \dots, x_n$ are roots of T_n.

Problem. *If $n \in \mathbb{N}$ and $f(x) \in \mathbb{R}^n[x]$ satisfies $|f(x)| \le \sqrt{1-x^2}$ $\forall x \in [-1,1]$, then $|f'(x)| \le 2(n-1)$ $\forall x \in [-1,1]$.*

SOLUTION. We prove three lemmas.

Lemma 1. *For any $P \in \mathbb{R}^{n-1}[x]$,*

$$P(x) = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}\sqrt{1-x_i^2}P(x_i)\frac{T_n(x)}{x-x_i}.$$

Proof. Lagrange's interpolation formula for $P(x)$ with nodes $x_1, x_2, \dots, x_n$ gives

$$P(x) = \sum_{i=1}^{n}\frac{P(x_i)}{\prod_{j\neq i}(x_i - x_j)}\prod_{j\neq i}(x-x_j) = \sum_{i=1}^{n}\frac{P(x_i)}{T_n'(x_i)}\cdot\frac{T_n(x)}{x-x_i}.$$

Since $T_n'(\cos\varphi)\sin\varphi = n\sin(n\varphi)$ $\forall \varphi \in \mathbb{R}$, we have

$$T_n'(x_i) = T_n'\left(\cos\left(\frac{2i-1}{2n}\pi\right)\right) = n\frac{\sin\left(\frac{2i-1}{2}\pi\right)}{\sin\left(\frac{2i-1}{2n}\pi\right)} = \frac{n(-1)^{i-1}}{\sqrt{1-x_i^2}} \quad \forall i = \overline{1,n}.$$

Hence

$$P(x) = \sum_{i=1}^{n}\frac{P(x_i)}{T_n'(x_i)}\cdot\frac{T_n(x)}{x-x_i} = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}\sqrt{1-x_i^2}P(x_i)\frac{T_n(x)}{x-x_i}. \qquad \square$$

Lemma 2. *If $P \in \mathbb{R}^{n-1}[x]$ satisfies $\sqrt{1-x^2}|P(x)| \le 1$ $\forall x \in [-1,1]$ then $|P(x)| \le n$ $\forall x \in [-1,1]$.*

Proof. We have $\sin(\frac{\pi}{2n}) \ge \frac{\pi}{2n}[1-(\frac{\pi}{2n})^2/6] \ge \frac{\pi}{2n}[1-(\frac{\pi}{4})^2/6] > \frac{1}{n}$ by the inequality $\sin x \ge x - \frac{x^3}{6}$ $\forall x \ge 0$. Therefore, if $|x| \le \cos(\frac{\pi}{2n}) = x_1$ then $\sqrt{1-x^2} \ge \sin(\frac{\pi}{2n}) > \frac{1}{n}$, hence $|P(x)| \le \frac{1}{\sqrt{1-x^2}} < n$.

Next, if $|x| > x_1$ then we can assume that $x > x_1$ (the case $x < -x_1$ can be done in the same manner). Since $x > x_1 > x_2 > \ldots > x_n$, $\frac{T_n(x)}{x-x_i}$ is positive for all $i = \overline{1,n}$. Combining this with Lemma 1 and the hypothesis $\sqrt{1-x^2}|P(x)| \le 1\ \forall x \in [-1,1]$, we obtain

$$\begin{aligned} |P(x)| &= \left| \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1} \sqrt{1-x_i^2} P(x_i) \frac{T_n(x)}{x-x_i} \right| \\ &\le \frac{1}{n} \sum_{i=1}^{n} \sqrt{1-x_i^2} |P(x_i)| \frac{T_n(x)}{x-x_i} \le \frac{1}{n} \sum_{i=1}^{n} \frac{T_n(x)}{x-x_i} = \frac{T_n'(x)}{n}. \end{aligned}$$

As $T_n'(x)$ attains local maximum in $(x_1, 1]$ at $x = 1$ and $T_n'(1) = nU_{n-1}(1) = n^2$, we deduce that $|P(x)| \le \frac{1}{n} T_n'(1) = n$, completing the proof of the lemma. □

Lemma 3. *If $g(t)$ is a trigonometric polynomial of degree at most n, or*

$$g(t) = \sum_{k=0}^{n} (a_k \cos(kt) + b_k \sin(kt)), \quad a_k, b_k \in \mathbb{R} \quad \forall k = \overline{1,n},$$

and $|g(t)| \le 1\ \forall t \in \mathbb{R}$, then $|g'(t)| \le n\ \forall t \in \mathbb{R}$.

Proof. We need to show that $|g'(t_0)| \le n$ for every fixed $t_0 \in \mathbb{R}$. Denote $h(t) := \frac{g(t_0+t)-g(t_0-t)}{2}$ $\forall t \in \mathbb{R}$. The identities

$$\begin{aligned} \cos(k(t_0+t)) - \cos(k(t_0-t)) &= -2\sin(kt_0)\sin(kt) \quad \forall t \in \mathbb{R}, k \in \mathbb{N}_0, \\ \sin(k(t_0+t)) - \sin(k(t_0-t)) &= -2\cos(kt_0)\sin(kt) \quad \forall t \in \mathbb{R}, k \in \mathbb{N}_0, \end{aligned}$$

assures that there are reals $b_1, b_2, \ldots, b_n$ satisfying

$$h(t) = b_1 \sin t + b_2 \sin(2t) + \cdots + b_n \sin(nt) = \sum_{k=1}^{n} b_k \sin(kt).$$

Because $\sin(kt) = U_{k-1}(\cos t)\sin t\ \forall t \in \mathbb{R}, k \in \mathbb{N}$, we have $h(t) = Q(\cos t)\sin t\ \forall t \in \mathbb{R}$ where $Q(x) \equiv \sum_{k=1}^{n} b_k U_{k-1}(x) \in \mathbb{R}^{n-1}[x]$. Furthermore, noting that $|h(t)| \le \frac{|g(t_0+t)|+|g(t_0-t)|}{2} \le 1$, we obtain $|Q(\cos t)\sin t| \le 1\ \forall t \in \mathbb{R}$. Letting $x := \cos t$, we get $\sqrt{1-x^2}|Q(x)| \le 1\ \forall x \in [-1,1]$. Thus by Lemma 2 $|Q(x)| \le n\ \forall x \in [-1,1]$ or $|Q(\cos t)| \le n\ \forall t \in \mathbb{R}$. This implies

$$|h'(0)| = \lim_{t \to 0} \left| \frac{h(t)}{t} \right| = \lim_{t \to 0} |Q(\cos t)| \cdot \lim_{t \to 0} \left| \frac{\sin t}{t} \right| \le n \cdot 1 = n,$$

but since $h'(0) = g'(t_0)$ as $h'(t) = \frac{g'(t_0+t)+g'(t_0-t)}{2}$ $\forall t \in \mathbb{R}$, we deduce that $|g'(t_0)| = |h'(0)| \le n$. The proof is completed. □

Now we are ready to prove the original problem. Since $|f(x)| \le \sqrt{1-x^2}$ $\forall x \in [-1,1]$, we have $f(-1) = f(1) = 0$, so $1-x^2 \mid f(x)$, and the problem is trivial if $n \le 1$. If $n \ge 2$, there exists $P(x) \in \mathbb{R}^{n-2}[x]$ such that $f(x) \equiv (1-x^2)P(x)$, which follows $\sqrt{1-x^2}|P(x)| \le 1$ $\forall x \in [-1,1]$. Hence, by Lemma 2 $|P(x)| \le n-1$ $\forall x \in [-1,1]$.

Next, let $x = \cos t$ for $t \in \mathbb{R}$, then $P(x) = P(\cos t) \in \mathbb{R}^{n-2}[\cos t]$ and $|P(\cos t)| = |P(x)| \le n-1$ $\forall t \in \mathbb{R}$. Since $\{\cos(kt) : k \in \{0,1,\dots,n-2\}\}$ is a basis for $\mathbb{R}^{n-2}[\cos t]$, there are unique reals a_k, $k = \overline{0,n-2}$ such that $P(\cos t) = \sum_{k=0}^{n-2} a_k \cos(kt)$. Denote $g(t) := P(\cos t)\sin t$ $\forall t \in \mathbb{R}$, then $f(\cos t) = g(t)\sin t$ $\forall t \in \mathbb{R}$. From the identity

$$2\cos(kt)\sin t = \sin((k+1)t) - \sin((k-1)t) \quad \forall t \in \mathbb{R}, k \in \mathbb{N}_0,$$

we see that $g(t)$ is a trigonometric polynomial of degree at most $n-1$. Moreover, because

$$|g(t)| = |P(\cos t)\sin t| = \sqrt{1-x^2}|P(x)| \le 1 \quad \forall t \in \mathbb{R},$$

by Lemma 3 we have $|g'(t)| \le n-1$ $\forall t \in \mathbb{R}$.

Finally, we shall compute the derivative of $f(\cos t)$ with respect to t in two ways. On the one hand, $\frac{\mathrm{d}f(\cos t)}{\mathrm{d}t} = -f'(\cos t)\sin t$, and on the other hand,

$$\frac{\mathrm{d}f(\cos t)}{\mathrm{d}t} = \frac{\mathrm{d}(g(t)\sin t)}{\mathrm{d}t} = g(t)\cos t + g'(t)\sin t = P(\cos t)\sin t\cos t + g'(t)\sin t.$$

Therefore, $-f'(\cos t) = P(\cos t)\cos t + g'(t)$, giving

$$\begin{aligned} |f'(\cos t)| &\le |P(\cos t)\cos t| + |g'(t)| \\ &= |P(\cos t)| \cdot |\cos t| + |g'(t)| \le (n-1)|\cos t| + (n-1) \le 2(n-1) \quad \forall t \in \mathbb{R}, \end{aligned}$$

or $|f'(x)| \le 2(n-1)$ $\forall x \in [-1,1]$. The problem is completely solved. ❑

Remark. Equality holds if $f(x) \equiv (1-x^2)U_{n-2}(x)$.